Identification for Control: A $\nu-$gap metric Approach

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Introduction

Questions Addressed

Given Two feedback loops : model-controller ('designed' loop), unknown true plant-controller ('achieved' loop)

1. Given frequency response samples of the true plant, **find** a model that approximates the plant in a suitable metric such that the designed loop $\approx$ the achieved loop for 'any' controller.

2. Given a bounded power excitation r, **characterise** the difference in the response of the designed and the achieved closed-loop to excitation r for 'any' controller.

Notation

G_i	normalised right graph symbol for $P_i = N_i M_i^{-1}$ ($G_i = \begin{bmatrix} N_i \\ M_i \end{bmatrix}$ is inner)
$\tilde{G}_i$	normalised left graph symbol for $P_i = \tilde{M}_i^{-1}\tilde{N}_i$ ($\tilde{G}_i = [-\tilde{M}_i \ \tilde{N}_i]$ is co-inner)
$\eta(P)$	number of unstable poles of P
wno	winding number; for $X, X^{-1} \in \mathcal{RL}_\infty$, wno det $(X) = \eta(X^{-1}) - \eta(X)$
$I(P_0, P_1)$	wno det $(G_1^* G_0)$
$\mathcal{R}^{m\times n}$	Finite dimensional LSI systems (m o/p, n i/p)

The ν–gap metric

The ν–gap between two plants P_0 and P_1 may be defined by

$$\begin{aligned}\delta_\nu(P_0, P_1) \quad &:= \inf_{\substack{Q, Q^{-1} \in \mathcal{L}_\infty \\ \text{wno}\det(Q)=0}} \| G_0 - G_1 Q \|_\infty \\ &:= \sup_\omega \kappa(P_0, P_1)(e^{j\omega}) \text{ when} \\ &\quad I(P_0, P_1) = 0\end{aligned}$$

The *pointwise chordal distance* $\kappa(P_0, P_1)$ defined by

$$\kappa(P_0, P_1)(e^{j\omega}) :=$$

$$\overline{\sigma}\left((I + P_0 P_0^*)^{-\frac{1}{2}}(P_0 - P_1)(I + P_1^* P_1)^{-\frac{1}{2}}\right)(e^{j\omega})$$

$$= \overline{\sigma}\,(\tilde{G}_2 G_1)\,(e^{j\omega})$$

The $\nu-$gap metric (cont'd)

Define closed loop transfer function for a plant P_i and controller C by

$$H(P_i, C) = \begin{bmatrix} P_i \\ I \end{bmatrix} (I - CP_i)^{-1} \begin{bmatrix} -C & I \end{bmatrix}$$

Fact: For a plant P_0, model P_1 and a controller C, at any point $e^{j\omega}$,

$$\begin{aligned} \kappa(P_0, P_1) &\leq \overline{\sigma}(H(P_0, C) - H(P_1, C)) \text{ and} \\ \overline{\sigma}(H(P_0, C) - H(P_1, C)) &\leq \alpha\kappa(P_0, P_1) \end{aligned} \qquad (1)$$

where $\alpha = \overline{\sigma}(H(P_1, C))\overline{\sigma}(H(P_0, C))$.

Hence, δ_ν(plant,model) is small $\Leftrightarrow$ difference in closed loop behaviour is small for any 'reasonable' controller

Identification Problem

Given: Frequency reponse samples of plant P_0,

$$P_0(e^{j\omega_i}), i = 1, 2, \ldots, m$$

A 'sensible' identification problem:

Find P_1 such that

$$\inf_{Q, Q^{-1} \in \mathcal{L}_\infty} \max_i \overline{\sigma}(\, G_0 - G_1 Q\,)(e^{j\omega_i})$$
$$= \max_i \kappa(P_0, P_1)(e^{j\omega_i})$$

is minimised; subject to a smoothness constraint on P_1 and $I(P_0, P_1) = 0$.

Strategy for an Approximate Solution

- Formulate an equivalent problem, in terms of $P_0(e^{j\omega_i})$ instead of $G_0(e^{j\omega_i})$;
- Choose a parameterised model set $\mathcal{S}$, solve

$$\min_{P \in \mathcal{S}} \max_i \kappa(P_0, P)(e^{j\omega_i})$$

  subject to some smoothness constraint on P. Let P_1 be the solution.

- Given a model P_1 and a controller C s.t. $\eta\left(H(P_0, C)\right) = 0$, find another model P_2 which satisfies $I(P_0, P_2) = 0$ and minimises $\max_\omega \kappa(P_1, P_2)$. Then

$$\max_i \kappa(P_0, P_2)(e^{j\omega_i}) \leq \max_i \kappa(P_0, P_1) + \max_\omega \kappa(P_1, P_2)$$

An Equivalent Problem Formulation

Lemma 1 *Given* $P_0, P_1 \in \mathcal{RL}_\infty$, $Q, Q^{-1} \in \mathcal{L}_\infty$, $\exists\ \hat{Q}, F$ *such that* $\hat{Q}$, $\hat{Q}^{-1}$, $F \in \mathcal{L}_\infty$,

$$\overline{\sigma}\left(G_0 - G_1 Q\right)\left(e^{j\omega}\right) = \overline{\sigma}\left(F - G_1\hat{Q}\right)\left(e^{j\omega}\right) \forall\, \omega$$

and at any ω, *complex matrix* $F(e^{j\omega})$ *can be written as a function of point frequency response matrix* $P_0(e^{j\omega})$.

For a SISO system, we can choose F as

$$F(e^{j\omega_i}) = \left[\frac{P_0(e^{j\omega_i})}{\sqrt{1+|P_0(e^{j\omega_i})|^2}} \ \frac{1}{\sqrt{1+|P_0(e^{j\omega_i})|^2}} \right]^T$$

Choice of Model Set (SISO case)

Let $S_n = \text{span}\{1,\, z,\, \ldots\,,\, z^{n-1}\}$ and

$$S_{1,2} = \{f : f = [f_1\ f_2]^T, f_1 \in S_{n_1},\ f_2 \in S_{n_2}\}$$

For $f \in S_{1,2}$, the constraints

$\max_i \overline{\sigma}(\,G_0 - fQ\,)(e^{j\omega_i}) < \lambda$ and

$\|f'\|_\infty < \lambda$ are affine in the parameters of f.

Chordal Distance Minimisation

Initialisation: Set $k = 1$, $\hat{Q}_0 = 1$.

Step A: Solve LMI optimisation

$$\min_{f_k \in S_{1,2}} \max_{i \in [1,m]} \left\{ \overline{\sigma}(F(e^{j\omega_i}) - f_k(e^{j\omega_i})\hat{Q}_{k-1}(e^{j\omega_i})) \; , \right. \\ \left. \alpha \| f_k' \|_\infty \right\}$$

where α is a user specified weight; $F(e^{j\omega_i})$ derived from $P_0(e^{j\omega_i})$. Let $\hat{f}_k$ be the solution.

Step B: Solve

$$\min_{Q_k(e^{j\omega_i})} \max_{i \in [1,m]} \overline{\sigma}(F(e^{j\omega_i}) - \hat{f}_k(e^{j\omega_i})Q_k(e^{j\omega_i}))$$

Let $\hat{Q}_k(e^{j\omega_i})$ be the solution. If $\max_i \overline{\sigma}(F(e^{j\omega_i}) - \hat{f}_k(e^{j\omega_i})\hat{Q}_k(e^{j\omega_i}))$ is less than a specified tolerance, stop; otherwise set $k := k + 1$ and go back to step A.

For $\hat{f}_k = [\hat{f}_1 \; \hat{f}_2]^T$, the model obtained is $P_1 = \hat{f}_1 \hat{f}_2^{-1}$.

Winding Number Adjustment - 1

Idea ***(Vinnicombe, 1993)*:**
Given P_1, P_2, C satisfying
$\kappa(P_1, P_2)(e^{j\omega})\overline{\sigma}\left(H(P_1, C)\right)\left(e^{j\omega}\right) < 1\, \forall\omega,$

$$\boxed{\eta\left(H(P_2, C)\right) = \eta\left(H(P_1, C)\right) + I(P_2, P_1)}$$

$\Rightarrow$ Given a controller C which stabilises plant P_0 and a model P_1 such that $\boxed{\eta\left(H(P_1, C)\right) = k}$

find another model P_2 such that

$\boxed{I(P_2, P_1) \leq -k}$

$$\Downarrow \text{ if } \kappa(P_1,P_2)(e^{j\omega})\overline{\sigma}(H(P_1,C))(e^{j\omega})<1\,\forall\,\omega$$

$\eta\left(H(P_2, C)\right) = 0$

$$\Downarrow \text{ if } \kappa(P_2,P_0)(e^{j\omega})\overline{\sigma}(H(P_2,C))(e^{j\omega})<1\,\forall\,\omega$$

$\boxed{I(P_2, P_0) = 0}$ (since $\eta\left(H(P_0, C)\right) = 0$)

Winding Number Adjustment - 2

Solution: Given P_1, solution of the chordal distance approximation problem and a controller C that stabilises P_0, Solve

$$\inf_{\substack{P \in \mathcal{R} \\ I(P,P_1) \leq -k}} \sup_{\omega} \kappa(P_1, P)(e^{j\omega})$$

where $k = \eta\,(H(P_1, C))$.

The solution to above problem may be characterised by solution of a Hankel norm approximation problem. If P_2 is a solution,

$$\max_i \kappa(P_0, P_2)(e^{j\omega_i}) \leq \max_i \kappa(P_0, P_1) + \max_{\omega} \kappa(P_1, P_2)$$

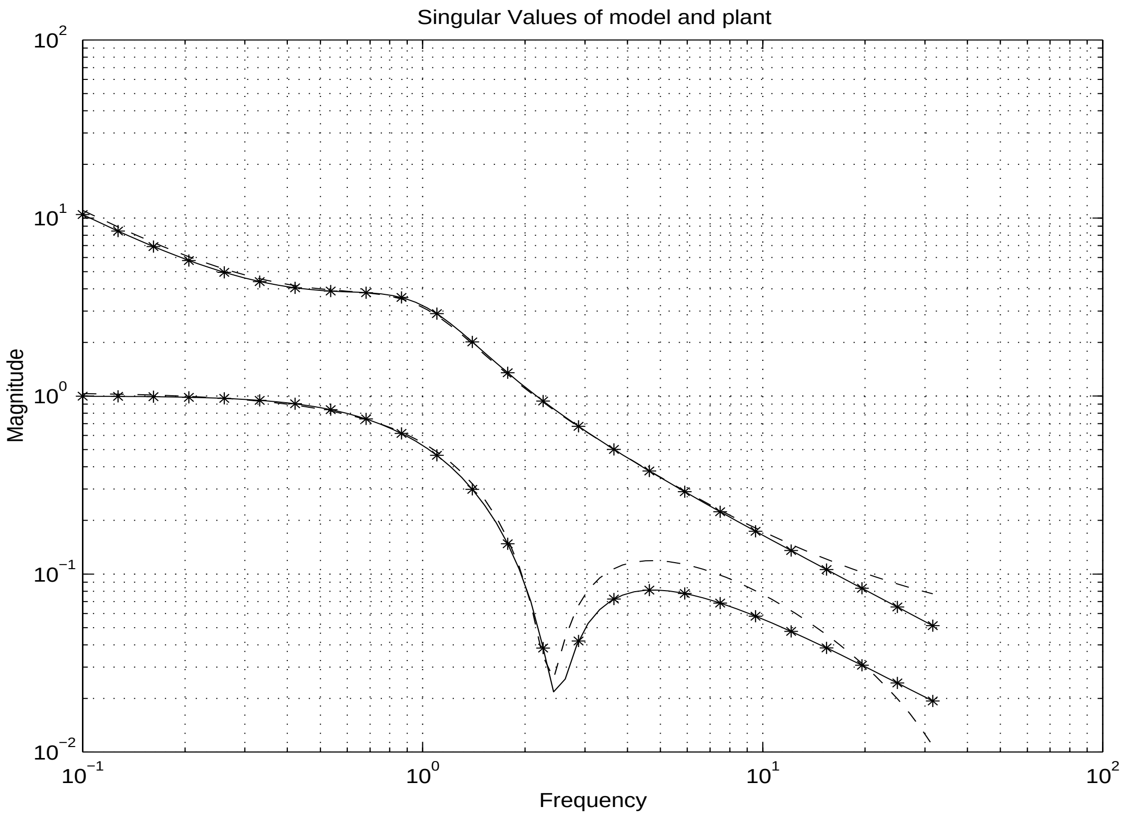
If $\eta\,(H(P_0, 0)) = \eta\,(H(P_1, 0)) = 0$, and $\max_i \kappa(P_0, P_1)(e^{j\omega_i})$ is 'small', winding number adjustment is not required (i.e. P_1 is our final model.).

Simulation Example

$$P_0(s) = \begin{bmatrix} \frac{1}{s+2} & \frac{1}{s} \\ \frac{1}{s-1} & \frac{3}{s^2+s+1} \end{bmatrix}$$

25 non-uniformly spaced freq. response samples. $\alpha = .0010$, order $= 3$ (each term).

Singular Value Plots



$$\max_i \kappa(\text{ plant, model})(e^{j\omega_i}) = 0.06 \approx \delta_\nu(\text{plant, model})$$

The Second Question

- Given a bounded power excitation r, **characterise** the difference in the response of the designed and the achieved closed-loop to excitation r for 'any' controller.

$$\kappa(P_0, P_1) \le \overline{\sigma}(H(P_0, C) - H(P_1, C)) \text{ and}$$
$$\overline{\sigma}(H(P_0, C) - H(P_1, C)) \le \alpha\kappa(P_0, P_1)$$

where $\alpha = \overline{\sigma}(H(P_1, C))\overline{\sigma}(H(P_0, C))$.

Does a similar result hold pointwise in the set of persistent signals?

Bounded Power Signals

- Let $R_u(\tau) = \lim_{N\to\infty} \frac{1}{N}\sum_{t=0}^{N-1} u(t-\tau)u^T(t)$.

- $$\mathcal{S}^n = \{u \,|\, u \in l_\infty^n(\mathbb{Z}),\ u(t) = 0 \,\forall\, t < 0,$$
$$R_u(\tau) \text{ exists } \forall\,\tau,\ \phi_u(\omega) := \sum_{\tau=-\infty}^{\infty} R_u(\tau)e^{-j\tau\omega}$$
$$\text{exists for all } \omega\}$$

- *Power spectrum* $\phi_u(\omega)$ need not be bounded and may contain impulses.

- $\|f\|_{\mathcal{S}} := \sqrt{\text{trace}\, R_f(0)} = \sqrt{\frac{1}{2\pi}\int_{-\pi}^{\pi} \text{trace}\, \phi_f(\omega)\, d\omega}.$

- Subset of interest:

$$\mathcal{S}^{n'} := \left\{v \,|\, v \in \mathcal{S}^n,\ \phi_v = \phi_{\tilde{v}} I_{n\times n} \text{ with } \tilde{v} \in \mathcal{S}^1\right\}$$

A New Measure of Distance over $\mathcal{R}^{m\times n}$

- Let $\mathcal{R}_2^{m\times n} := \{(P_1, P_2) : P_1, P_2 \in \mathcal{R}^{m\times n}\}$

  Define $\delta_r : \mathcal{R}_2^{m\times n} \times \mathcal{S}^{n'} \to \mathbb{R}$

$$\delta_r(P_1, P_2, r) := \frac{\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} \text{trace } (\tilde{G}_2 G_1)\phi_r(\tilde{G}_2 G_1)^* d\omega\right\}^{\frac{1}{2}}}{\|r\|_{\mathcal{S}}}$$

$$= \frac{\|\,\tilde{G}_2 G_1\, r\,\|_{\mathcal{S}}}{\|r\|_{\mathcal{S}}}$$

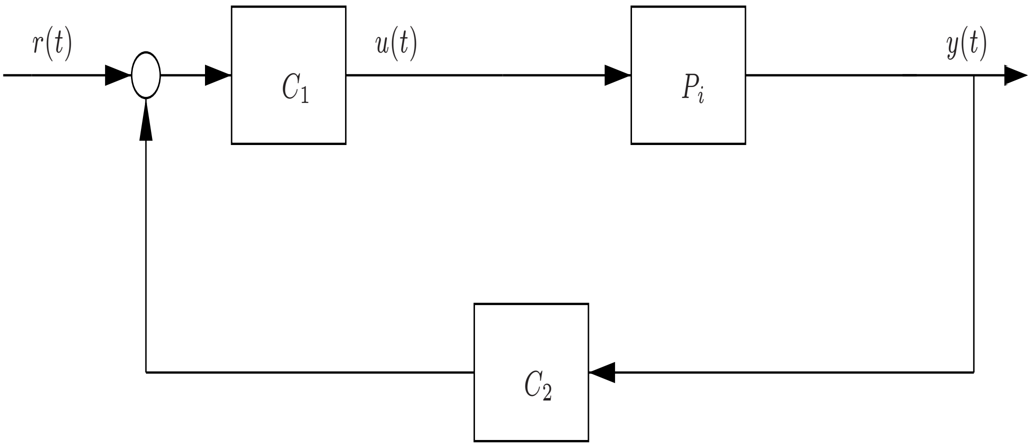
- $0 \leq \delta_r(P_1, P_2, r) \leq 1.$

- For any $r_o \in \mathcal{S}^{n'}$ such that $\phi_{r_0} \neq 0\, \forall\, \omega \in [-\pi, \pi]$, $\delta_r(P_1, P_2, r_o)$ is a metric.

- For $n = 1$,

$$\sup_{r\in\mathcal{S}^{1'}} \delta_r(P_1, P_2, r) = \sup_{\omega} \kappa(P_1, P_2) =: \delta_{\mathcal{L}_2}(P_1, P_2)$$

Closed Loop Set-up



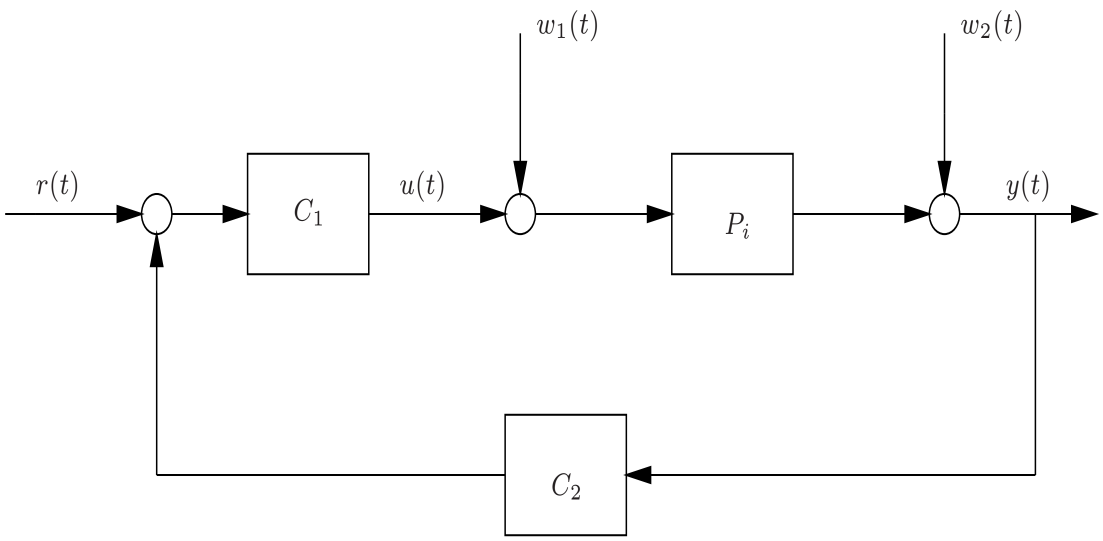
- C_1 is such that $\underline{\sigma}\left(C_1\right)\left(e^{j\omega}\right) > 0$.

- $T(P_i, C_1, C_2)$ = transfer function from r to $\begin{bmatrix} y \\ u \end{bmatrix}$.

- $b(P_i, C) = \|H(P_i, C)\|_\infty^{-1}$ ('generalised robust stability margin'; larger the better).

Bounds on δ_r

Data Generating System



- **Given:** Measured data $z := \begin{bmatrix} y \\ u \end{bmatrix}$, power $\| r \|_{\mathcal{S}}$.

- r, w_i such that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{t=0}^{N-1} (r(t) w_i^T(t-\tau)) = 0 \; \forall \, \tau, i = 1, 2$$

Example 1

- $P_1(z) = \frac{2(z+1)}{z-0.6}$, $P_2(z) = \frac{(z+1)^2}{(z^2-0.6z+1.2)}$.
- $\delta_\nu(P_1, P_2) = 0.64$.
- If $\phi_{r_0} = f(e^{j\omega})f(e^{-j\omega})$, $f(z) = \frac{0.01z}{z-0.99}$,

  $\delta_r(P_1, P_2, r_0) = 0.0417$.
- $C_1 = \frac{0.025(z+1)}{(z-1)}$, $C_2 = -1$

  $$\Downarrow$$

  $b(P_1, C) = 0.408$, $b(P_2, C) = 0.401$.

Example 2

- $P_3(z) = \frac{z-1}{z-0.99}$, $P_4(z) = \frac{z-1}{z-1.01}$.
- $\delta_\nu(P_1, P_2) = 1$.
- If $\phi_{r_0} = f(e^{j\omega})f(e^{-j\omega})$, $f(z) = 1$,

  $\delta_r(P_1, P_2, r_0) = 0.055$.
- $C_1 = 1$, $C_2 = -1$

  $$\Downarrow$$

  $b(P_1, C) = b(P_2, C) = 0.7071$.

Summary

- A new algorithm for frequency domain identification; identification strategy to minimise the difference in the closed loop behaviour between the model and the plant for *any* controller.
- A new measure of distance for systems under bounded power excitation; time domain characterisation of difference in closed loop behaviour under persistent reference/disturbances.

Finally,

Related Work *(by other people)*

- Bombois, Gevers, Scorletti, ECC'99: Given ϵ, X, $\hat{\theta}$,

  $$\Phi = \{\theta \;:\; (\hat{\theta} - \theta)^T X (\hat{\theta} - \theta) < \epsilon\}$$

  Solve $\min_{\theta \in \Phi} \; \kappa(P_{\hat{\theta}}, P_\theta)$.

- Steele, Vinnicombe, submitted to CDC'01: Given $\hat{P}$, $\{y_N, u_N\}$, ϵ, δ,

  $$\mathcal{P} = \{P : \|y_N - P u_N\|_q \leq \delta, \delta_\nu(P, \hat{P}) \leq \epsilon\}$$

  Is $\mathcal{P}$ non-empty?