Weight Selection and Identification for $\mathcal{H}_\infty$ Loop-shaping

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Introduction

Question Addressed

Given:

- Frequency response samples of the true plant
- Frequency domain closed loop specifications

Find: A model that approximates the plant in a suitable norm, such that
(behaviour of the plant in closed loop with *any* controller C which satisfies the given closed loop spec.s) $\approx$ (behaviour of the model in closed loop with same controller)

$\mathcal{H}_\infty$ Loop-shaping

(McFarlane and Glover, 1990, 1992)

Define

$H(P,C)$ closed loop transfer function,

$$= \begin{bmatrix} P \\ I \end{bmatrix} (I - CP)^{-1} \begin{bmatrix} -C & I \end{bmatrix}$$

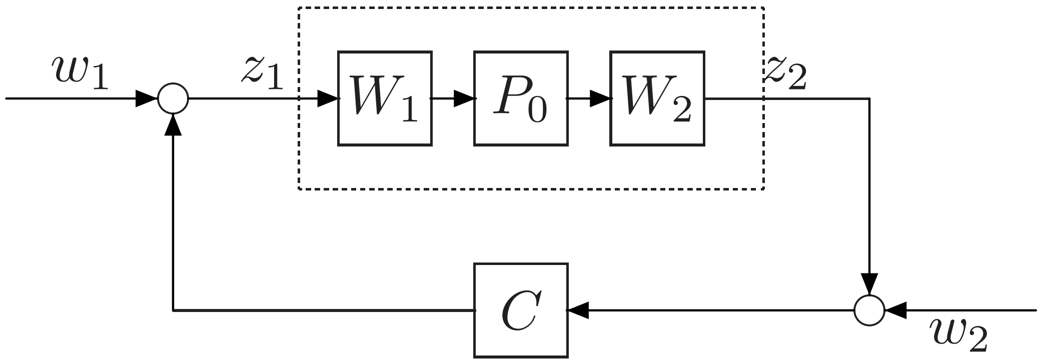
$b(P,C)$ generalised robust stability margin,

$= \|H(P,C)\|_\infty^{-1}$ if $H(P,C)$ is stable

$= 0$ otherwise.

$\rho(P,C)$ 'pointwise $b(P,C)$',

$$= \frac{1}{\overline{\sigma}(H(P,C))}$$



$\mathcal{H}_\infty$-Loop-shaping: Setup

- $$\begin{bmatrix} z_2 \\ z_1 \end{bmatrix} = H(P_s, C) \begin{bmatrix} w_2 \\ w_1 \end{bmatrix}$$
- W_1, W_2 are (usually) min. phase, diagonal transfer matrices such that $P_s = W_2 P_0 W_1$ has desired loopshape (*e.g.* large $\underline{\sigma}(P_s)$ in some low freq. region, small $\overline{\sigma}(P_s)$ in some high freq. region).

$\mathcal{H}_\infty$-Loop-shaping: Procedure

- Choose weights W_1, W_2 such that, for given loopshape bounds γ_1, γ_2,

$$\gamma_1(\omega) \leq \sigma_i(W_2 P_0 W_1)(j\omega) \leq \gamma_2(\omega)$$

  holds for all ω.

- Solve

$$\sup_{C \text{ stabilising}} \quad b(W_2 P_0 W_1, C)$$

  (This has an analytic solution: McFarlane and Glover, 1990). Let C_∞ be the solution.

- Final controller is $C_s = W_1 C_\infty W_2$.

$\mathcal{H}_\infty$-Loop-shaping: Properties

- **Performance:** If the chosen loopshape ($\sigma_i(P_s)$) is compatible to robust stability requirement (*i.e.* if $b_{opt}(P_s)$ is large), $\sigma_i(P_0C_s) \approx \sigma_i(P_s)$. W_1, W_2 are for performance; C_∞ - ideally - only adjusts phase for stability.
- **Robustness:** $b(P_s, C_\infty)$ represents robustness to normalised coprime factor perturbations; W_1, W_2 decide the shape of acceptable perturbations.

Question: If $b(P_s, C_\infty) > \beta$, what is the largest set of plants which is stabilised by C_∞?

The ν–gap metric

Define

G_i	normalised right graph symbol for $P_i = N_i M_i^{-1}$ ($G_i = \begin{bmatrix} N_i \\ M_i \end{bmatrix}$ is inner)
$\eta(P)$	number of unstable poles of P
wno	winding number; for $X, X^{-1} \in \mathcal{RL}_\infty$, wno det $(X) = \eta(X^{-1}) - \eta(X)$
$I(P_0, P_1)$	wno det $(G_1^* G_0)$

The ν−gap metric: Definition

(Vinnicombe, 1993)

The ν−gap between two plants P_0 and P_1 may be defined by

$$\begin{aligned} \delta_\nu(P_0, P_1) \;&:=\; \inf_{\substack{Q,Q^{-1}\in\mathcal{L}_\infty \\ \text{wno}\det(Q)=0}} \| G_0 - G_1 Q \|_\infty \\ &:=\; \sup_\omega \kappa(P_0, P_1)(j\omega) \text{ when} \\ &\qquad I(P_0, P_1) = 0 \end{aligned}$$

The *pointwise chordal distance* $\kappa(P_0, P_1)$ defined by

$$\kappa(P_0, P_1)(j\omega) :=$$

$$\overline{\sigma}\left((I + P_0 P_0^*)^{-\frac{1}{2}}(P_0 - P_1)(I + P_1^* P_1)^{-\frac{1}{2}}\right)(j\omega)$$

The $\nu-$gap metric: Properties

- For a plant P_0, model P_1 and a controller C, at any point ω,

$$\kappa(P_0, P_1) \leq \overline{\sigma}(H(P_0, C) - H(P_1, C)) \text{ and}$$

$$\overline{\sigma}(H(P_0, C) - H(P_1, C)) \leq \frac{\kappa(P_0, P_1)}{\rho(P_0, C)\,\rho(P_1, C)}$$

- Also,

$$\rho(P_0, C) \geq \rho(P_1, C) - \kappa(P_0, P_1) \text{ and}$$

$$b(P_0, C) \geq b(P_1, C) - \delta_\nu(P_0, P_1) \;\Leftarrow\; \text{tight bound}$$

- If P_1 is a model for plant P_0,
  - ◦ $\rho(P_1, C)$: designed performance.
  - ◦ $\rho(P_0, C)$: achieved performance.
  - ◦ $\kappa(P_0, P_1)$: difference in closed loop behaviour (**no C in this.)**

Weight Selection and Identification Problem

Given: Frequency reponse samples of plant P_0, $P_0(j\omega_i), i = 1, 2, \ldots, m$ and loopshaping specifications $\gamma_{1,i}$, $\gamma_{2,i}$:

Find:

- Diagonal, stable min. phase t.f.s W_1, W_2 s.t.

$$\boxed{\gamma_{1,i} \;\leq\; \sigma_i(W_2 P_0 W_1)(j\omega_i) \;\leq\; \gamma_{2,i}}$$

- $\hat{P} \in \mathcal{S}$ for a suitable $\mathcal{S}$ which solves

$$\boxed{\min_{P \in \mathcal{S}} \max_i \kappa(W_2 P_0 W_1, P)(j\omega_i)}$$

subject to a smoothness constraint on $\hat{P}$ and the constraint $I(W_2 P_0 W_1, \hat{P}) = 0$.

Weight Selection: Convex Feasibility Approach (Lanzon, 2000)

- Find diagonal, positive definite X_i, Y_i such that

  $$\gamma_{1,i}\, X_i \;\leq P_0(j\omega_i)^* Y_i P_0(j\omega_i) \;\leq \gamma_{2,i}\, X_i$$

  holds. Then

  $$\gamma_{1,i} \;\leq \sigma_i(Y_i^{\frac{1}{2}} P_0(j\omega_i) X_i^{-\frac{1}{2}}) \;\leq \gamma_{2,i}$$

- Find minimum phase W_1, W_2 s.t. $W_1^{-*}W_1^{-1}(j\omega_i) \;\approx\; X_i, W_2^{*}W_2(j\omega_i) \;\approx\; Y_i$.
- It is possible to include further, useful convex constraints, *e.g.* bounds on condition numbers of X_i, Y_i.

Identification Problem

Given: Frequency reponse samples of shaped plant $P_s = W_2 P_0 W_1$,

$$P_s(j\omega_i), i = 1, 2, \ldots, m$$

Find $P_1 \in \mathcal{S}$ such that

$$\inf_{Q, Q^{-1} \in \mathcal{L}_\infty} \max_i \overline{\sigma}(\, G_s - G_1 Q \,)(j\omega_i)$$
$$= \max_i \kappa(P_s, P_1)(j\omega_i)$$

is minimised; subject to a smoothness constraint on P_1 and $I(P_s, P_1) = 0$.

Strategy for an approximate solution

(Date-Vinnicombe 1999,2001)

- Formulate an equivalent problem, in terms of $P_s(j\omega_i)$ instead of $G_s(j\omega_i)$;
- Choose a parameterised model set $\mathcal{S}$, solve

$$\min_{P \in S} \max_i \kappa(P_s, P)(j\omega_i)$$

  subject to some smoothness constraint on P. Let P_1 be the solution.

- Given a model P_1 and a controller C s.t. $\eta\left(H(P_s, C)\right) = 0$, find another model P_2 which satisfies $I(P_s, P_2) = 0$ and minimises $\max_\omega \kappa(P_1, P_2)$. Then

$$\max_i \kappa(P_s, P_2) \leq \max_i \kappa(P_s, P_1) + \max_\omega \kappa(P_1, P_2)$$

An Equivalent Problem Formulation

Lemma 1 *Given* $P_s, P_1 \in \mathcal{RL}_\infty$, $Q, Q^{-1} \in \mathcal{L}_\infty$, $\exists\ \hat{Q}, F$ *such that* $\hat{Q}$, $\hat{Q}^{-1}$, $F \in \mathcal{L}_\infty$,

$$\overline{\sigma}\left(G_s - G_1 Q\right)(j\omega) = \overline{\sigma}\left(F - G_1\hat{Q}\right)(j\omega)\ \forall\, \omega$$

and at any ω, *complex matrix* $F(j\omega)$ *can be written as a function of point frequency response matrix* $P_s(j\omega)$.

For a SISO system, we can choose F such that

$$F(j\omega_i) = \left[\frac{P_s(j\omega_i)}{\sqrt{1+|P_s(j\omega_i)|^2}}\ \frac{1}{\sqrt{1+|P_s(j\omega_i)|^2}}\right]^T$$

Choice of Model Set (SISO case)

Let $S_n = \text{span}\{1,\, x,\, \ldots,\, x^{n-1}\}$ where x is a fixed, first order transfer function. Let

$$S_{1,2} = \{f : f = [f_1\ f_2]^T, f_1 \in S_{n_1},\ f_2 \in S_{n_2}\}$$

For $f \in S_{1,2}$, the constraints

$\max_i \overline{\sigma}(\, G_s - fQ\,)(j\omega_i) < \lambda$ and

$\|f'\|_\infty < \lambda$ are affine in the parameters of f.

Chordal Distance Minimisation

Initialisation: Set $k = 1$, $\hat{Q}_0 = 1$.

Step A: Solve LMI optimisation

$$\min_{f_k \in S_{1,2}} \max_{i \in [1,m]} \Big\{ \overline{\sigma}(F(j\omega_i) - f_k(j\omega_i)\hat{Q}_{k-1}(j\omega_i)) \,, \\ \alpha \| f_k' \|_\infty \Big\}$$

where α is a user specified weight; $F(j\omega_i)$ derived from $P_s(j\omega_i)$. Let $\hat{f}_k$ be the solution.

Step B: Solve

$$\min_{Q_k(j\omega_i)} \max_{i \in [1,m]} \overline{\sigma}(F(j\omega_i) - \hat{f}_k(j\omega_i)Q_k(j\omega_i))$$

Let $\hat{Q}_k(j\omega_i)$ be the solution. If $\max_i \overline{\sigma}(F(j\omega_i) - \hat{f}_k(j\omega_i)\hat{Q}_k(j\omega_i))$ is less than a specified tolerance, stop; otherwise set $k := k + 1$ and go back to step A.

For $\hat{f}_k = [\hat{f}_1 \; \hat{f}_2]^T$, the model obtained is $P_1 = \hat{f}_1 \hat{f}_2^{-1}$.

Winding Number Adjustment - 1

Idea *(Vinnicombe, 1993)*:
Given P_1, P_2, C satisfying
$\kappa(P_1, P_2)(j\omega)\overline{\sigma}\left(H(P_1, C)\right)(j\omega) < 1\,\forall\omega$,

$$\boxed{\eta\left(H(P_2, C)\right) = \eta\left(H(P_1, C)\right) + I(P_2, P_1)}$$

$\Rightarrow$ Given a controller C which stabilises shaped plant P_s and a model P_1 such that

$$\boxed{\eta\left(H(P_1, C)\right) = k}$$

find another model P_2 such that

$$\boxed{I(P_2, P_1) \leq -k}$$

$\Downarrow$ if $\kappa(P_1,P_2)(j\omega)\overline{\sigma}(H(P_1,C))(j\omega)<1\,\forall\,\omega$

$$\eta\left(H(P_2, C)\right) = 0$$

$\Downarrow$ if $\kappa(P_2,P_0)(j\omega)\overline{\sigma}(H(P_2,C))(j\omega)<1\,\forall\,\omega$

$\boxed{I(P_2, P_s) = 0}$ (since $\eta\left(H(P_s, C)\right) = 0$)

Winding Number Adjustment - 2

Solution: Given P_1, solution of the chordal distance approximation problem and a controller C that stabilises P_s, Solve

$$\inf_{\substack{P\in\mathcal{R} \\ I(P,P_1)\leq -k}} \sup_{\omega} \kappa(P_1, P)(j\omega)$$

where $k = \eta\,(H(P_1, C))$.

The solution to above problem may be characterised by solution of a Hankel norm approximation problem. If P_2 is a solution,

$$\max_i \kappa(P_s, P_2)(j\omega_i) \leq \max_i \kappa(P_s, P_1) + \max_\omega \kappa(P_1, P_2)$$

If $\eta\,(H(P_s, 0)) = \eta\,(H(P_1, 0)) = 0$, and $\max_i \kappa(P_s, P_1)(j\omega_i)$ is ‘small’, winding number adjustment is not required (i.e. P_1 is our final model.).

Weight and Model Re-adjustment

(Date-Lanzon, 2001)

Idea:

- Let $\hat{P}$ be a model for plant P. For any compatible W_1, W_2,

$$\rho(W_2 P W_1, -(W_1^{-1} \hat{P} W_2^{-1})^{\sim}) = \sqrt{1 - \kappa^2(W_2 P W_1, W_1^{-1} \hat{P} W_2^{-1})}$$

- So

$$\min_{W_1, W_2} \kappa(W_2 P W_1, W_1^{-1} \hat{P} W_2^{-1})$$

is equivalent to

$$\max_{W_1, W_2} \rho(W_2 P W_2, -(W_1^{-1} \hat{P} W_2^{-1})^{\sim})$$

Weight and Model Re-adjustment

Procedure:

- **Given:** $P_0(j\omega_i)$, model P_2 and weights $W_{1,0}, W_{2,0}$.
- Set $\overline{P} = -(W_{1,0} P_2^{\sim} W_{2,0})^{\sim}$.
- Solve, subject to constraints on $W_2 P_0 W_1$,

  $$\max_{W_1,\, W_2} \; \rho(W_2 P_0 W_1, W_1^{-1} \overline{P}^{\sim} W_2^{-1})$$

  This is convex in $W_1^{-*} W_1^{-1}$ and $W_2^* W_2$ (Lanzon, 2000).
- New weights: $W_{2,1}, W_{1,1}$, result of this optimisation.
  New model: $P_3 = -(W_{2,1}^{-1} \overline{P}^{\sim} W_{2,1}^{-1})^{\sim}$.

  $$\kappa(W_{2,1} P_0 W_{1,1}, P_3) \le \kappa(W_{2,0} P_0 W_{1,0}, P_2)$$

Effect of Noise

Suppose, noisy freq. response samples,

$$P_{\omega_i} = P_0(j\omega_i) + v(j\omega_i),\ i = 1, \ldots, m$$

are given.

- Solve (approximately)

$$\min_{W_1, W_2, P} \max_i \kappa(W_2 P_{\omega_i} W_1, P)$$

  Let $\hat{P}, \hat{W}_1, \hat{W}_2$ be the solutions.

- Then

$$\begin{aligned}\kappa(\hat{W}_2 P_0 \hat{W}_1, \hat{P}) \leq\ & \kappa(\hat{W}_2 P_0 \hat{W}_1, \hat{W}_2 P_{\omega_i} \hat{W}_1) \\ & + \kappa(W_2 P_{\omega_i} W_1, \hat{P})\end{aligned}$$

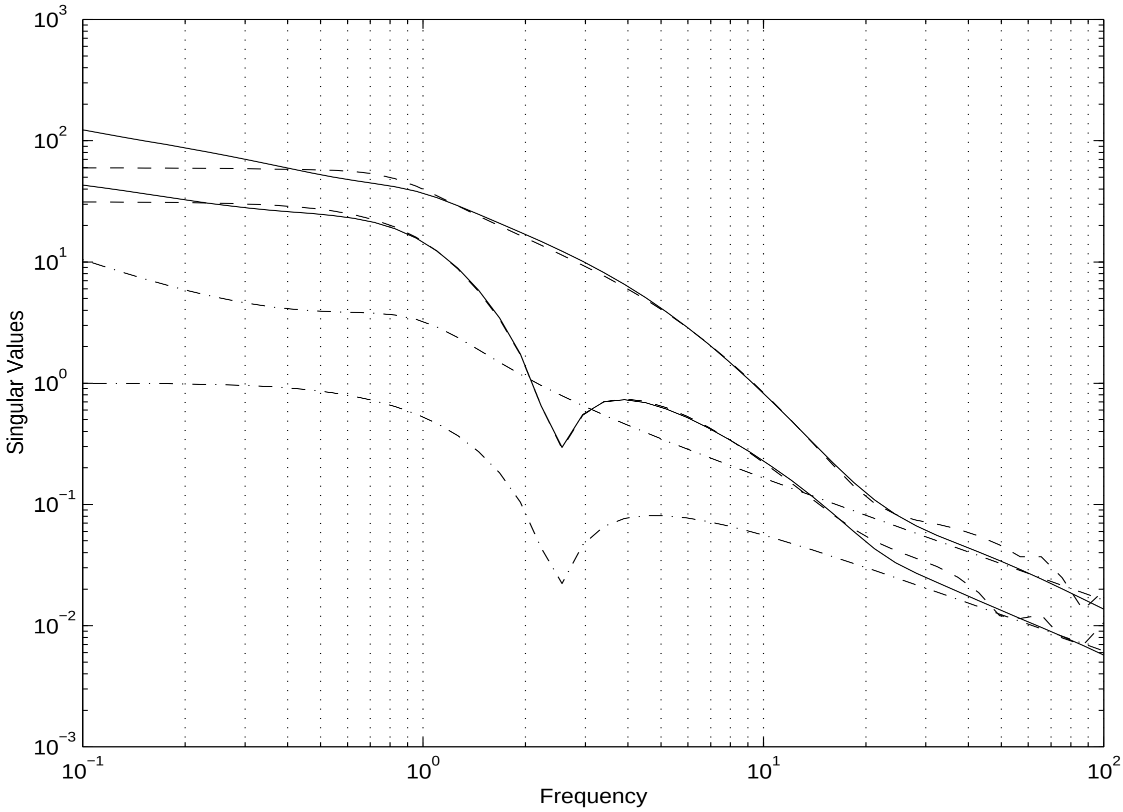
- First term is small if noise around closed loop crossover is small.

Simulation Example

- $P_0(s) = \begin{bmatrix} \frac{1}{s+2} & \frac{1}{s} \\ \frac{1}{s-1} & \frac{3}{s^2+s+1} \end{bmatrix}$
- 50 log-spaced freq. response samples bet$^{\text{n}}$

  $0.1\, rad/s$ and $100\, rad/s$.
- $\gamma_{1,i} \ \geq\ 25\, \forall\, \omega_i \ \leq\ .5$.
- $\gamma_{2,i} \leq .1\, \forall\, \omega_i \ \geq\ 20$.
- On solving feasibility problem and fitting transfer functions to pointwise weights: $W_1(s)$ of order 6, $W_2(s)$ of order 2.
- α chosen as .0010, model order = 18 ($n = 9$).

Singular Value Plots-1

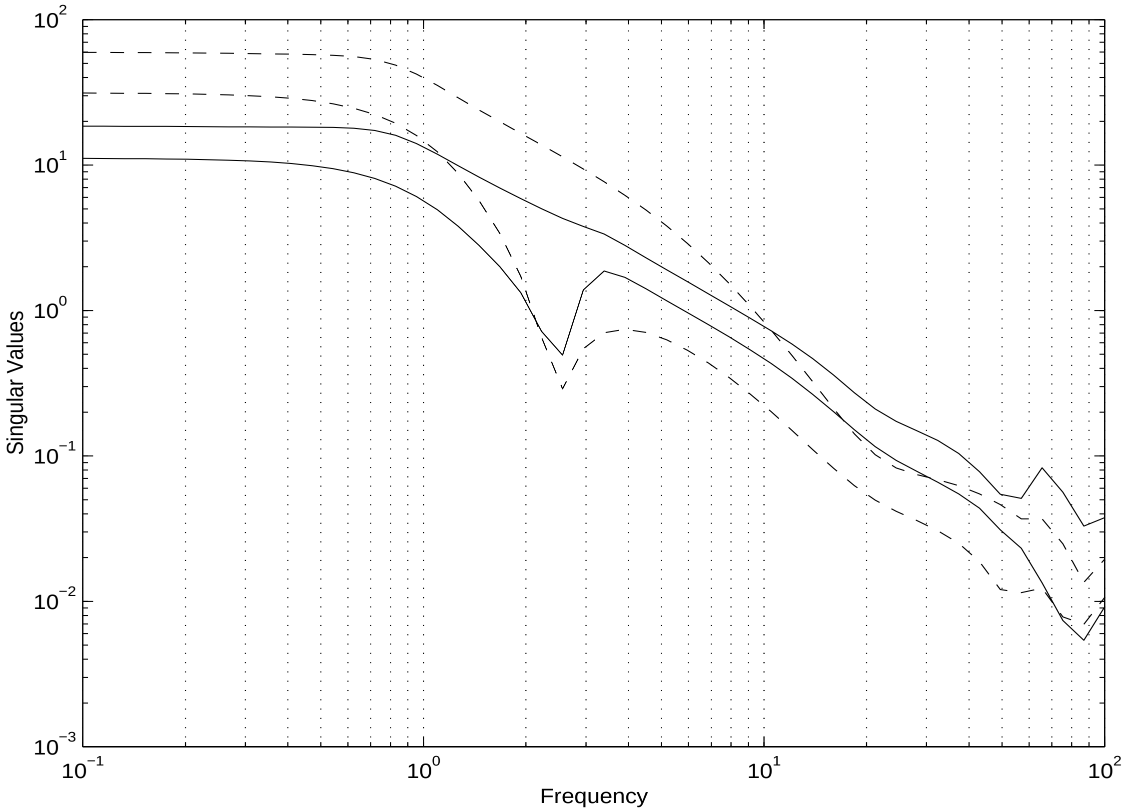


(achieved $\max_i \kappa(W_2 P_0 W_1, P_2)(j\omega_i) = 0.0126$,

true $\delta_\nu(W_2 P_0 W_1, P_2) = 0.037$)

Singular Value Plots-2

Open and closed loop-shape for the model



Summary

- For a weighted plant $P_s = W_2P_0W_1$, model P_2 and a controller C, at any point ω,

$$\rho(P_s, C) \geq \rho(P_2, C) - \kappa(P_s, P_2) \text{ and}$$
$$b(P_s, C) \geq b(P_2, C) - \delta_\nu(P_s, P_2)$$

- Weights W_1, W_2 can be obtained for given specifications using solution of convex feasibility problem.
- If chosen loop-shape is sensible and $\delta_\nu(P_s, P_2)$ is small, $b_{opt}(P_2)$ is large.
- $\max_i\ \kappa(P_s, P)$ explicitly minimised using a three step identification algorithm.
- Given P_2, it is possible to re-adjust weights W_1, W_2 and model P_2 to reduce $\max_i \kappa(W_2P_0W_1, P)(j\omega_i)$ further.

Related Work

1. Date, Vinnicombe 2002: Given a bounded power excitation r, characterise the difference in the response of the designed and the achieved closed-loop to excitation r for 'any' controller.

2. Steele, Vinnicombe 2001: Given $\hat{P}$, $\{y_N, u_N\}$, ϵ, δ,

   $$\mathcal{P} = \{P : \underbrace{\|y_N - Pu_N\|_q}_{(\textit{something similar})} \leq \delta, \delta_\nu(P, \hat{P}) \leq \epsilon\}$$

   Is $\mathcal{P}$ non-empty?