# Optimum decision making under uncertainty

(Stochastic Programming)

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# Outline

- Stochastic Programming in a nutshell
  - Two-stage models
  - Multi-stage models
  - Chance constrained models
- Simple financial application (SMPL)
- The SPInE software environment
- Discussion

# Linear Program

**We consider first a linear programming problem:**

$$\begin{aligned} Z &= \min\ cx \\ subject\ \ to\ \ Ax &= b \\ x &\geq 0 \end{aligned} \qquad (1)$$

$$where \quad A \in R^{m \times n};\ c, x \in R^{n};\ b \in R^{m}$$

**Deterministic optimisation model inapplicable to optimum decision making under uncertainty**

# Random Parameters

**Let (Ω, P) denote a probability space, $\omega \in \Omega$ the realizations of the uncertain parameters and p(ω) the corresponding probability. Let us denote the realizations of A, b, c for a given ω as:**

$$(A, b, c)_\omega = \xi_\omega \quad or \quad \xi(\omega).$$

**Let:**

$$C^\omega = \{x \mid Ax = b, x \geq 0\} \quad for \quad (A, b, c)_\omega \ or \ \xi(\omega)$$

**We can reconsider (1) as an expected value or an average value problem where**

$$\bar{\xi}(\omega) = E[\xi(\omega)] = \sum_{\omega \in \Omega} p(\omega)\xi(\omega)$$

$$Z_{ev} = \min f(x, \bar{\xi}) = \min \bar{c}x \qquad (2)$$

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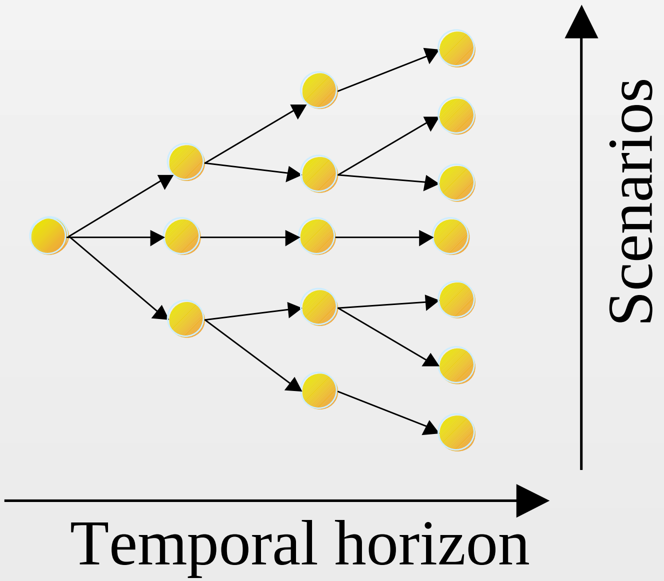
# Stochastic Programming (SP)

- Mathematical Programming Problem
- Uncertainty in the parameters

- Concept of scenarios
- Dimensions



# Classical Stochastic Linear Program with Recourse

**The SLPR is stated as**

$$
\begin{aligned}
Z = \ & \min && cx + E_{\omega}Q(x,\omega) \\
& subject\ \ to && Ax \qquad\qquad\quad = b \qquad (3) \\
& && x \geq 0,
\end{aligned}
$$

**where**

$$
\begin{aligned}
Q(x,\omega) = \ & \min && f(\omega)y \\
& subject\ \ to && D(\omega)y = d(\omega) + B(\omega)x \qquad (4) \\
& && y \geq 0.
\end{aligned}
$$

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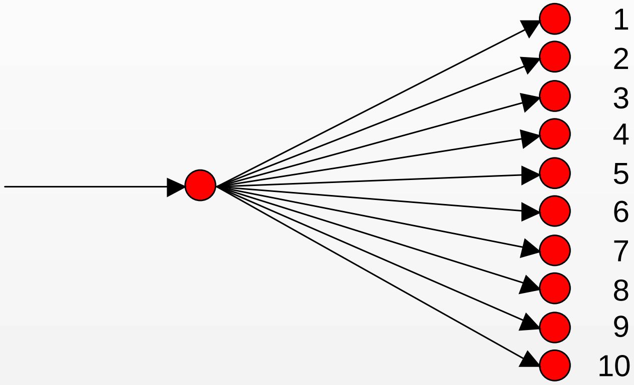
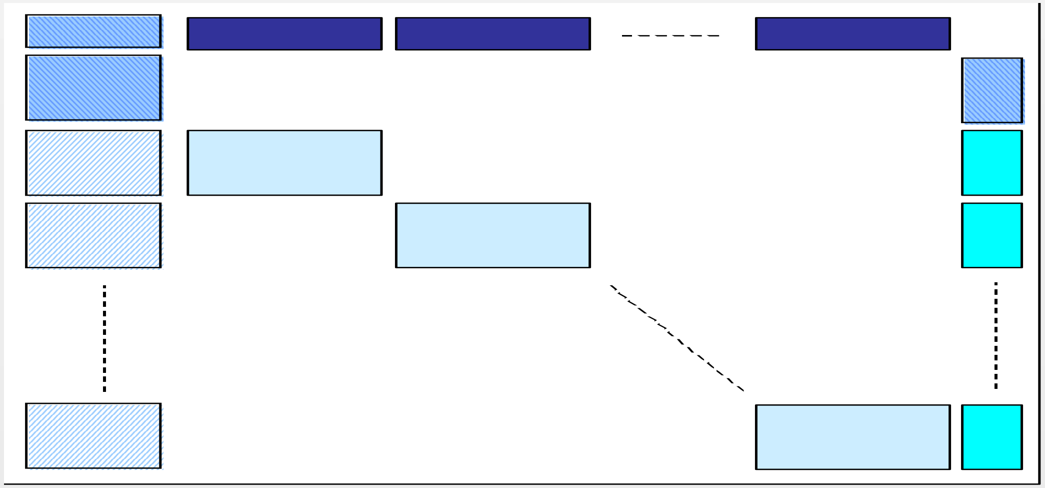
# Two Stage Stochastic Linear Programs

**A two-stage recourse problem is transformed into a two-stage stochastic linear program:**

$$
\begin{aligned}
\min Z \quad & = cx + E^{\omega}\left[ f y^{\omega} \right] \\
\textit{subject to} \quad & Ax = b \\
& -B^{\omega}x + D^{\omega}y^{\omega} = d^{\omega} \\
& x, y^{\omega} \geq 0; \quad \omega \in \Omega
\end{aligned}
\qquad (5)
$$

# Two-Stage Recourse Problem (TSRP)



# (Deterministic) Equivalent LP

**Let us consider the case of $\Omega$ being discrete and finite,and $\Omega$ as an index set $\Omega=\{1,...,K\}$, meaning that the parameter $\omega$ may take on K different values:**

$$
\begin{array}{llll}
\min Z & = cx + p^1 f y^1 + p^2 f y^2 + \ldots + p^K f y^K & & \\
\textit{subject to} & Ax & = b & \\
& -B^1 x + D^1 y^1 & = d^1 & \\
& -B^2 x + \qquad D^2 y^2 & = d^2 & (6) \\
& \vdots \qquad\qquad \ddots & \vdots & \\
& -B^K x + \qquad\qquad D^K y^K & = d^K & \\
& x, \quad y^1, \quad y^2, \quad \ldots \quad y^K \geq 0 & & \\
& 0 \leq p^\omega \leq 1 \quad \textit{and} \quad \sum p^\omega = 1.0 & &
\end{array}
$$

# Multiperiod Linear Model

**A general dynamic linear programming problem over the time horizon [1...T] and is formulated mathematically as:**

$$
\begin{array}{lll}
\min & c_1 x_1 + c_2 x_2 + \quad \ldots \quad + c_T x_T & \\
\textit{subject to} & A_{11} x_1 & = b_1 \\
& A_{21} x_1 + A_{22} x_2 & = b_2 \\
& A_{31} x_1 + A_{32} x_2 + A_{33} x_3 & = b_3 \qquad (7) \\
& \vdots \qquad \ddots & \vdots \\
& A_{T1} x_1 + A_{T2} x_1 + A_{T3} x_3 + \ldots + A_{TT} x_T & = b_T \\
& \ell_t \le x_t \le u_t ; t = 1, \ldots, T &
\end{array}
$$

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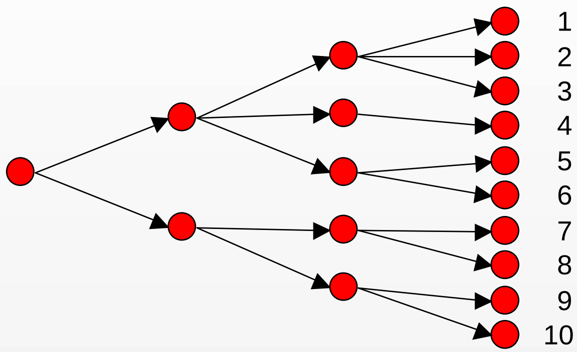
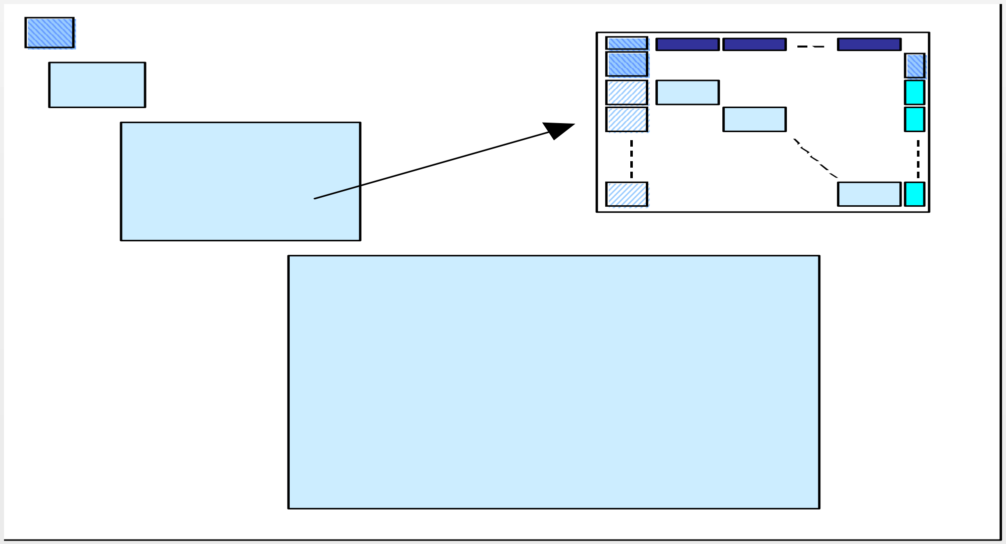
# Multistage Stochastic Programs (MSP)

$$\min_{x_1} \left\{ c_1 x_1 + E_{\xi_2}\left[ \min_{x_2} c_2 x_2 + E_{\xi_3|\xi_2}\left[ \min_{x_3} c_3 x_3 + \ldots + E_{\xi_T|\xi_{T-1}|\ldots|\xi_2} \min_{x_T} c_T x_T \right]\right]\right\}$$

$$\begin{array}{llllll} st & A_{11}x_1 & & & & = b_1 \\ & A_{21}x_1 + & A_{22}x_2 & & & = b_2 \\ & A_{31}x_1 + & A_{32}x_1 + & A_{33}x_3 & & = b_3 \\ & \vdots & & & \ddots & \vdots \\ & A_{T1}x_1 + & A_{T2}x_1 + & A_{T3}x_3 + & \ldots & + A_{TT}x_T = b_T \\ & \ell_t \le x_t \le u_t;\ t = 1,\ldots,T & & & & \end{array} \quad (8)$$

# Multi-Stage Recourse Problem (MSRP)



# Expectation of the Expected Value Problem

**Let $x^*_{ev}$ be the optimum solution of the expected value problem (2). This solution can be evaluated for all possible scenarios $\omega \in \Omega$ and we can determine the corresponding objective function values and compute the expected value of the objective function as $Z_{eev}$.**

$$Z_{eev} = E\left[cx^*_{ev}\right]$$

**Note c is a component of $\xi$. In particular we note that $x^*_{ev}$ may not be feasible for all $c^{\omega}$, that is for some $\omega$, $x^*_{ev} \notin C^{\omega}$; in this case we set $Z_{eev} \rightarrow +\infty$.**

# “Wait-and-See” Approach

**The corresponding problem is stated as:**

$$Z^{\omega} = \min \; cx \qquad (9)$$

$$\text{subject to} \quad x \in C^{\omega}$$

**Let $Z_{ws}$ denote the expected value of $Z^{\omega}$, then**

$$Z_{ws} = E\left[Z^{\omega}\right] = \sum_{\omega \in \Omega} Z^{\omega} p(\omega) \qquad (10)$$

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# "Here-and-Now" Decision Problem

**The value *x* is chosen such that the expected costs E(cx) assume a minimum:**

$$Z_{hn} = \min E[cx] \qquad (11)$$

$$where \quad x \in C$$

$$and \quad C = \bigcap_{\omega \in \Omega} C^{\omega} \qquad (12)$$

**The optimal objective function value $Z_{hn}$ denotes the minimum expected costs of the stochastic optimization problem.**

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# Inter Relationship and Bounds

**The three solutions $Z_{ws}$, $Z_{hn}$, $Z_{eev}$ are connected by the following ordered relationship.**

$$Z_{ws} \leq Z_{hn} \leq Z_{eev} \qquad (13)$$

**The difference ($Z_{hn}$ – $Z_{ws}$) is known as the expected value of perfect information (EVPI). Thus**

$$EVPI = Z_{hn} - Z_{ws} \qquad (14)$$

**There is another measure known as the value of the stochastic solution (VSS) which is defined as**

$$VSS = Z_{eev} - Z_{hn} \qquad (15)$$

**Birge shows how the EVPI and VSS can be bounded. These bounds are given as**

$$0 \leq EVPI \leq Z_{hn} - Z_{ev} \leq Z_{eev} - Z_{ev}$$

$$0 \leq VSS \leq Z_{eev} - Z_{ev}$$

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# Chance Constrained Programs (1)

**Consider the $i^{th}$ restriction**

$$\sum_j \overline{a}_{ij} x_j \le \overline{D}_i \qquad (16)$$

**We interpret this as satisfying the $i^{th}$ restriction with a probability $P_I$,**

$$P_i\left(\sum_j \overline{a}_{ij} x_j \le \overline{D}_i\right) = 1 \qquad (17)$$

**that is, the restriction is always satisfied. We can study the distribution of $a_{ij}$ or that of $\overline{D}_i$ and introduce the chance constraint**

$$P_i\left(\sum_j \overline{a}_{ij} x_j \le \overline{D}_i\right) \ge \beta_i \qquad (18)$$

**where typically $0.95 \le \beta_i \le 1$; and we interpret that the constraint (16) is satisfied with probability $\beta_i$ .**

# Chance Constrained Programs (2)

**If we consider the probability distribution of $D_i$ then we can find a value $B_i$ which is the $(1-\beta_i)$ fractile of the probability distribution, that is,**

$$P_i(\ D_i \geq B_i\ ) \geq \ \beta_i$$

**Having found $B_i$ we can replace each occurrence of (16) by the linear restriction**

$$\sum \bar{a}_{ij} x_j \leq B_i \qquad (19)$$

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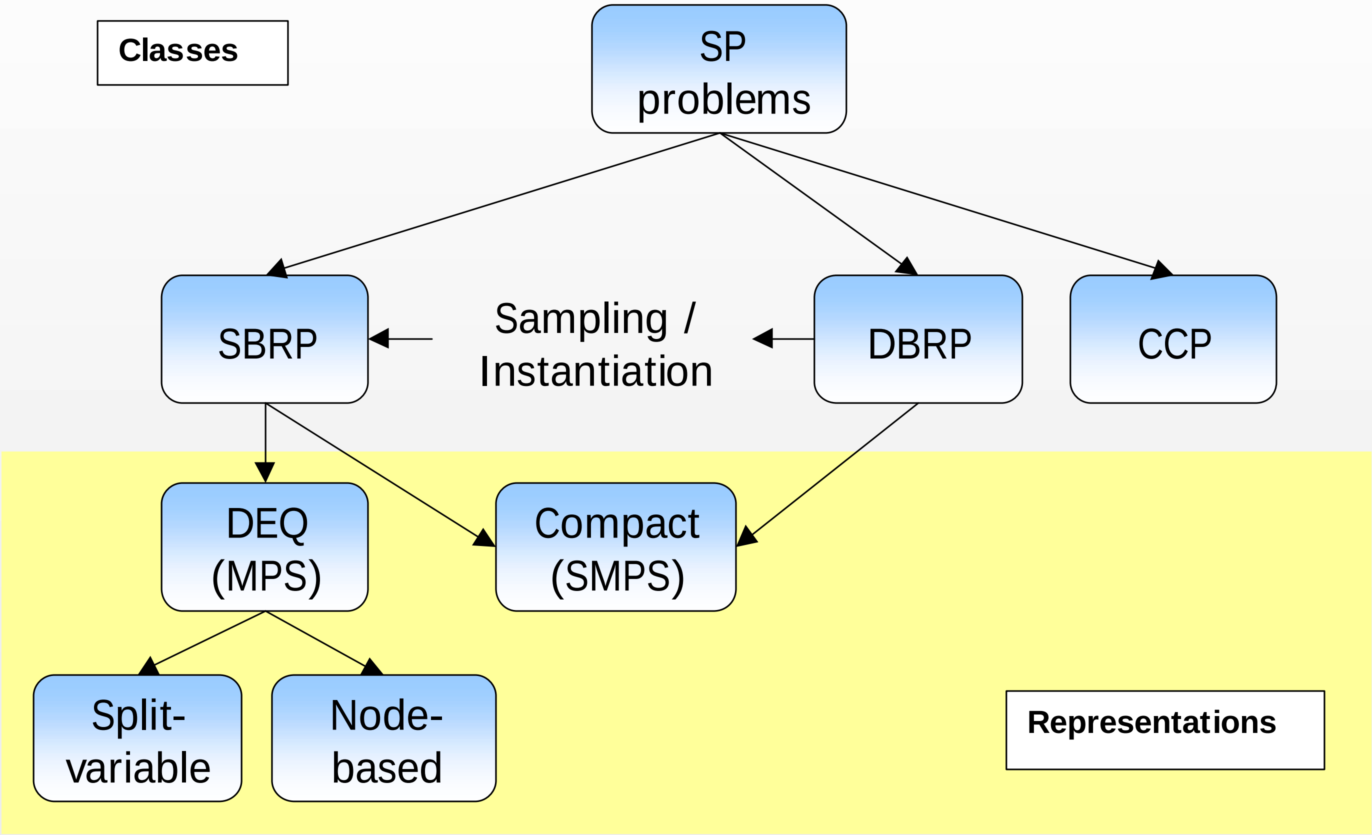
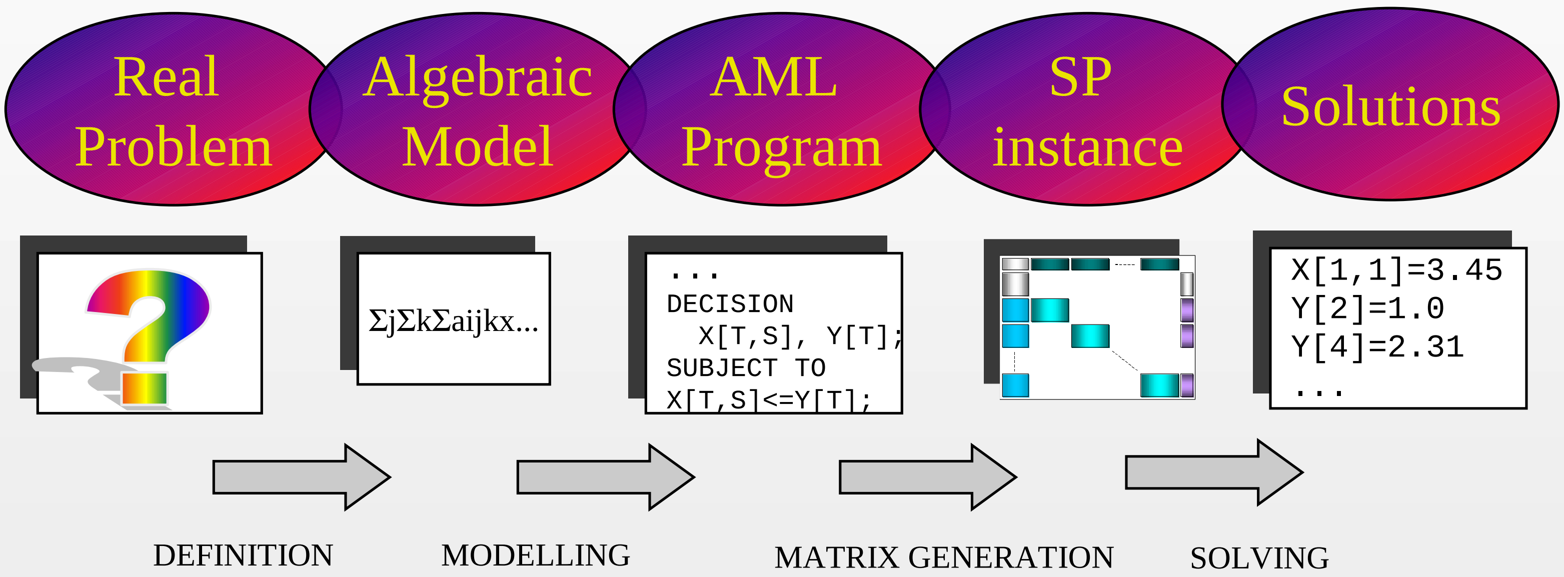
# Classification of SP problems



# Modelling and solving (1)



# Modelling SP (1)

- Current AMLs can only be used to define and generate deterministic equivalent of SBRP.
- Split-variable representations require *non-anticipativity* constraints
- Node-based representations are difficult to implement and maintain

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# Extending AMLs for SP (1)

- Describe SP models naturally by exploiting the inherent properties of their random structure
- Keep model and data instances separated
- Eliminate redundancy in the generated matrix

# Extending AMLs for SP (2)

- Consider SP models as refinement of deterministic problems by introduction of uncertainty

- SP models identify:
  - An underlying deterministic model
  - Some sort of stochastic information

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# SBRP stochastic info

- Random coefficients (parameters)
- Data paths weights (probabilities)
- Structure of the scenario tree
- Stages information (aggregation)
- Time structure

# DBRP stochastic info

- Time structure
- Random coefficients and their probability distributions
- Stages information (aggregation)
- Sampling information (optional)

# CCP stochastic info

- Random coefficients
- Probability distribution of the random coefficients
- Identification of individual and joint chance constraints

# Example

```
TITLE GasPerTutti

STOCHASTIC
          RPSB;

TIME
          T=(0,1);!
SCENARIO
          Scen=1..3;

PROBABILITIES
          PROB[Scen] = UNIFORM;

STAGES
1:        0..0;
2:        1..1;

TREE
          TWOSTAGE;

RANDOM DATA
D[T,Scen]  =
FILE("demand.dat"); !Gas demand

C[T,Scen]=
FILE("Cost.dat"); Cost per unit of gas
```

```
DATA
          S[T] = (0,0);   !Storage costs

DECISION
          Buy[T] ->X;
          Store[T]  ->Y;
          StorageLevel[T] ->Z;
MODEL
          MIN SUM(T: C*X)+SUM(T:S*Z);

SUBJECT TO

Satisfy[T]:
          X[T]+Y[T]>=D[T];

Balance[T] WHERE (T>0):
          Z[T]=Z[T-1]+X[T]-D[T];

Balance[T] WHERE (T=0):
          Z[T]=X[T]-D[T];

Use[T] WHERE (T>0):
          Y[T]<=Z[T-1];

Use[T] WHERE (T=0):
          Y[T]=0;
```

# EVPI and VSS

| Year | Buy | Store | Inventory | Cost |
|---|---|---|---|---|
| 0 Normal | 100 | 100 | 100 | 1100 |
| 1 Normal | 0 | 0 | 0 | 0 |
| 1 Cold | 50 | 0 | 0 | 300 |
| 1 Very cold | 80 | 0 | 0 | 600 |

$$Z_{eev} = \frac{1}{3}(1403.33 + 1400 + 1635) = 1479.44$$

$$Z_{hn} = 1100 + \frac{1}{3}(0 + 300 + 600) = 1400$$

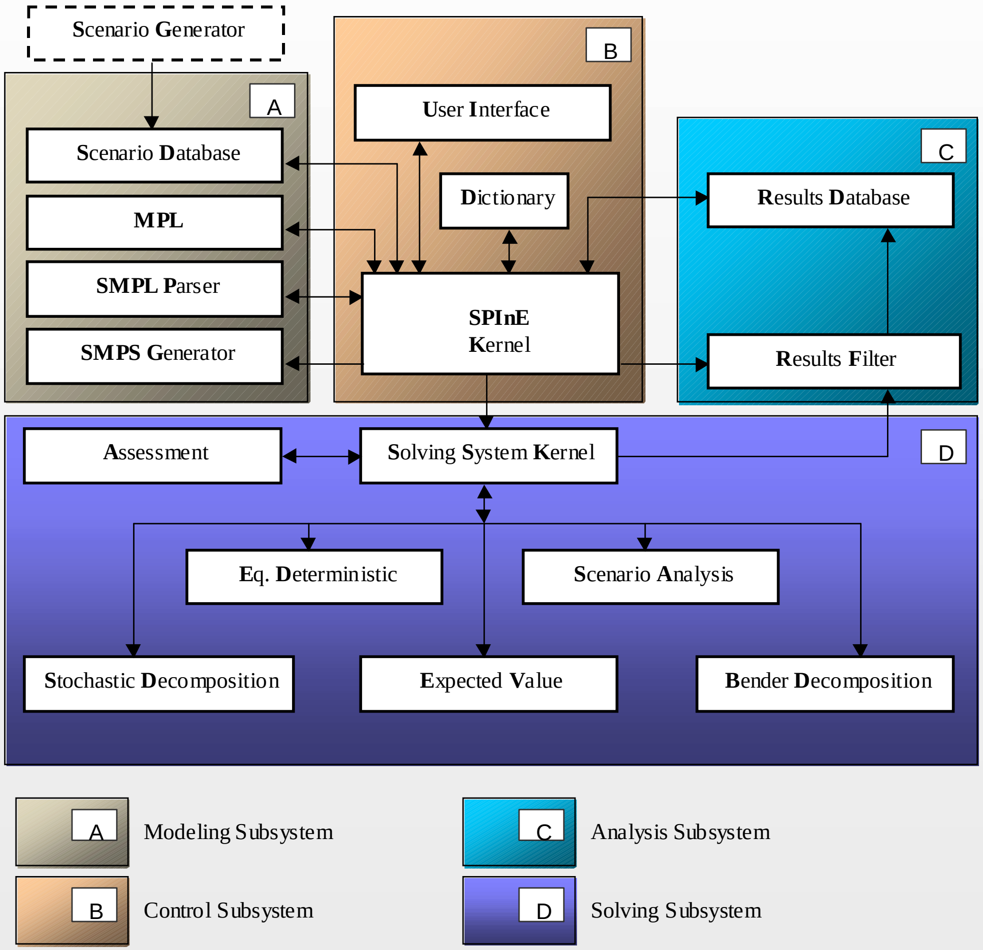
$$Z_{ws} = \frac{1}{3}(1000 + 1400 + 1580) = 1326.67$$

- **EVPI = $ 73.33**
- **VSS = $ 79.44 (5.4%)**

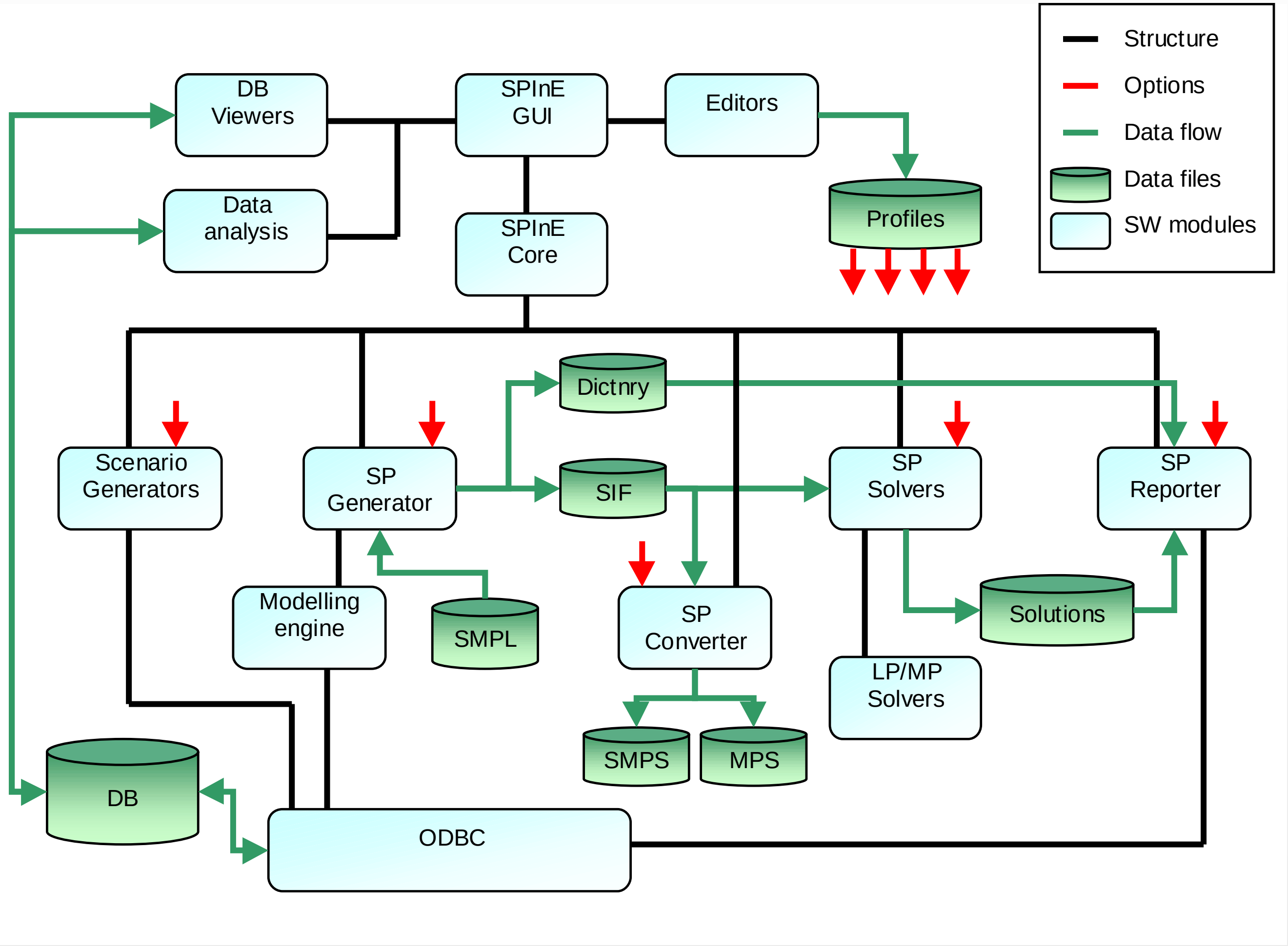
# SPInE

- Stochastic Programming Integrated Environment
- Windows 95/NT based
- supports SMPL/SAMPL
- Integration with Database through ODBC connection
- Integration with Scenario Generators

# SPInE Architecture (systems)



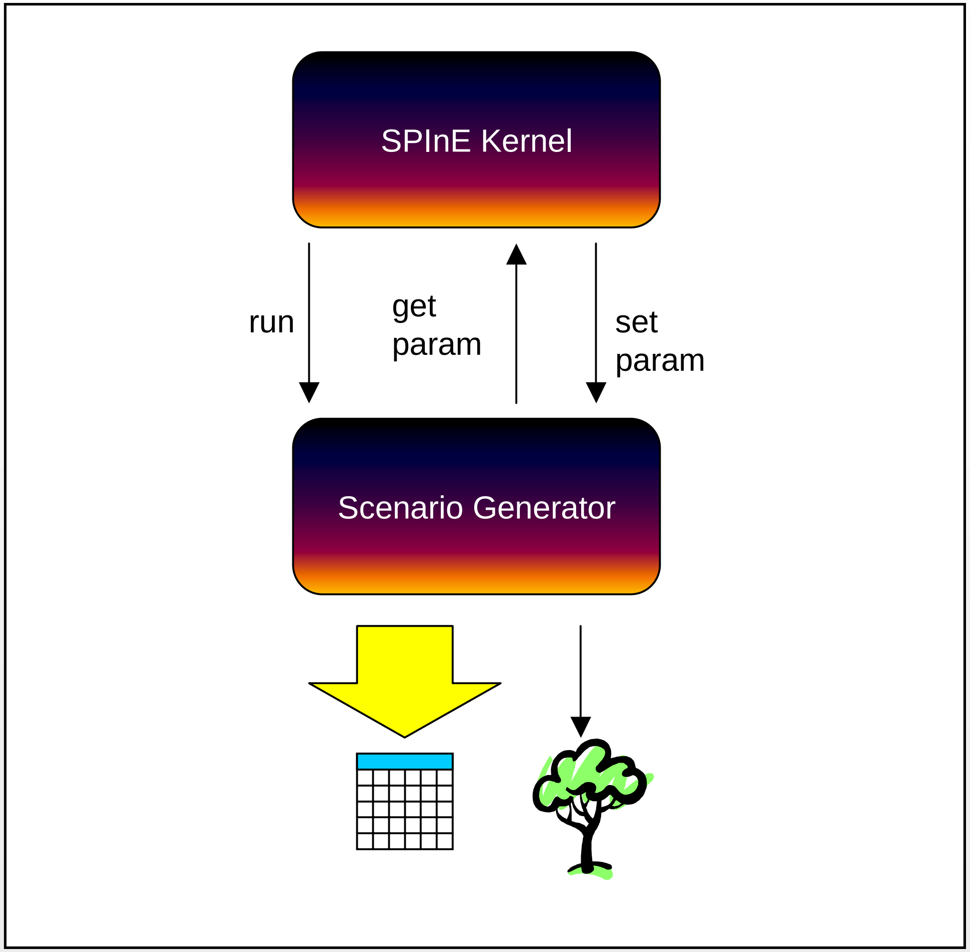
# SPInE Architecture (components)



# Scenario generator

- Highly specialised applications
- Any scenario generator can be attached
- Control communication protocol
- Database interface
- “Standard” simulation/sampling routines are provided

# Scenario generator



# Modelling subsystem

- Support of MPL/AMPL/Optimax2000 modelling engines
- SMPL/SAMPL pre-processor
- SPG (Stochastic Program Generator)

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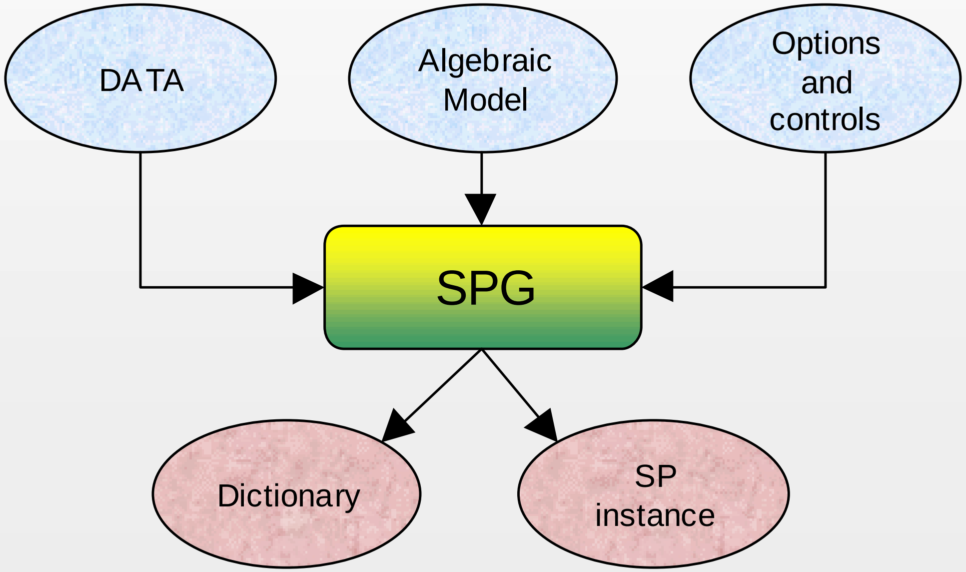
# SPG module

- *Instantiation* of a SMPL/SAMPL model
- Compact matrix output



# SP instance generation

- Iterative Process:
  - 1. Extract Stochastic Information
  - 2. Create underlying deterministic AM template
  - 2. Create sub-instance for current scenario
  - 3. Update global instance
  - 4. Back to 2. Until all scenarios have been processed

# Modelling subsystem

- Support of MPL/AMPL/Optimax2000 modelling engines
- SMPL/SAMPL pre-processor
- SPG (Stochastic Program Generator)

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# Solving Subsystem

- Three *strategies:*
  - Scenario Analysis Solution
  - Expected Value Solution
  - Stochastic Solution (Here and now)
- Based on FortMP
- SMPS and SIR input
- Any other SP solver can be used thanks to SMPS

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# Solver

- Embedded Algorithms:
  - Scenario Analysis
  - Expected Value
  - (*Nested) Benders Decomposition
  - Deterministic Equivalent (Simplex & Interior Point)
  - Lagrangean relaxation & importance sampling

# Analysis Subsystem

- Database analysis tools & viewers
  - Enables Advanced Analysis
  - Simplifies deduction of relevant information
- Solution Filter (Reporter)
  - Reduces redundancies on decisions

# Control Subsystem

- User Interface
  - Deep Interaction with subsystems
  - Based on concept of *project*
    - Models and Data Instances management
    - Simplifies maintenance and investigation
    - Allow validation/evaluation

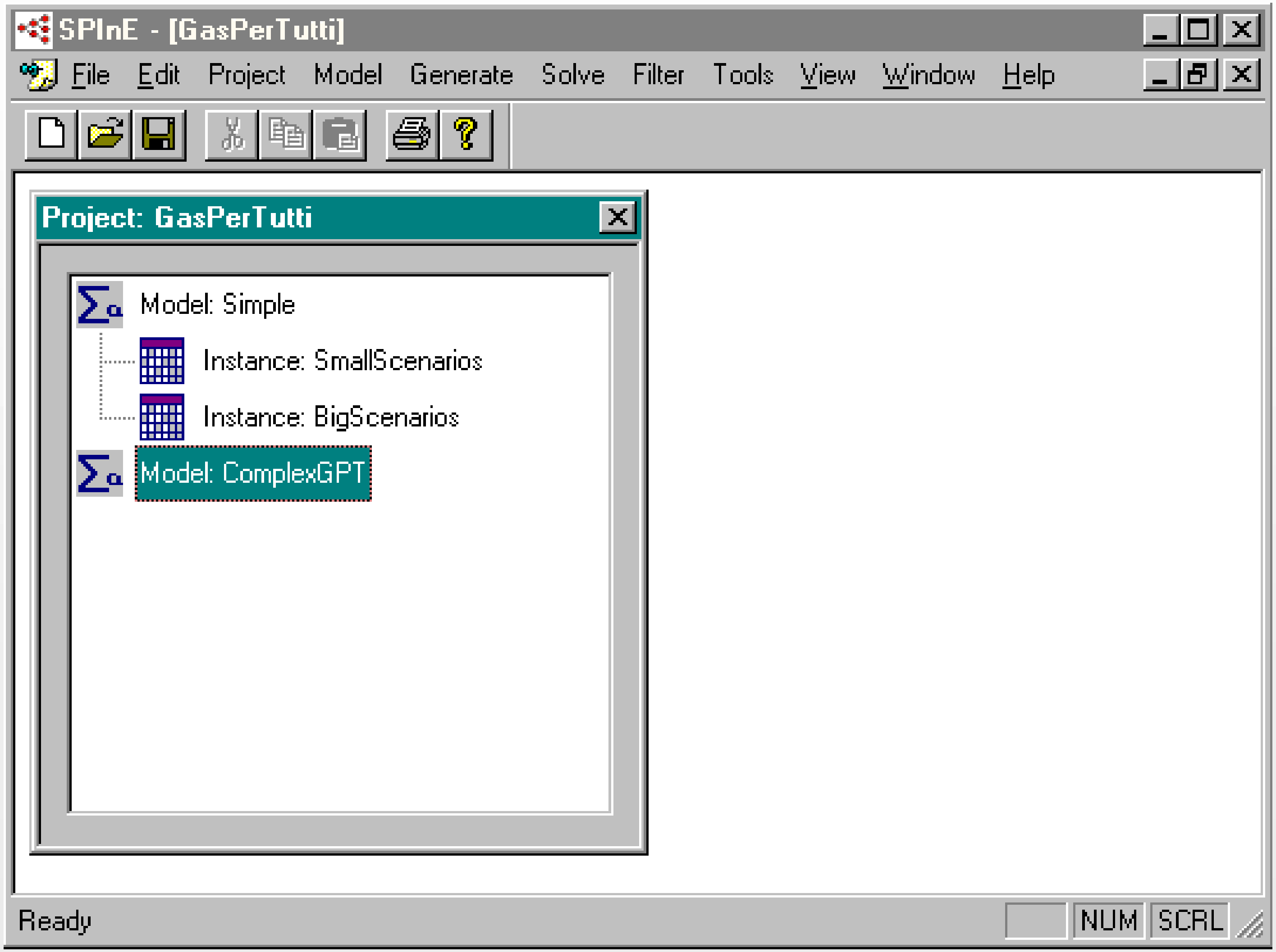
# Control Subsystem

- SPInE Kernel
  - Controls and drives the information flow of the application
- Dictionary
  - Translation of variable and constraint names

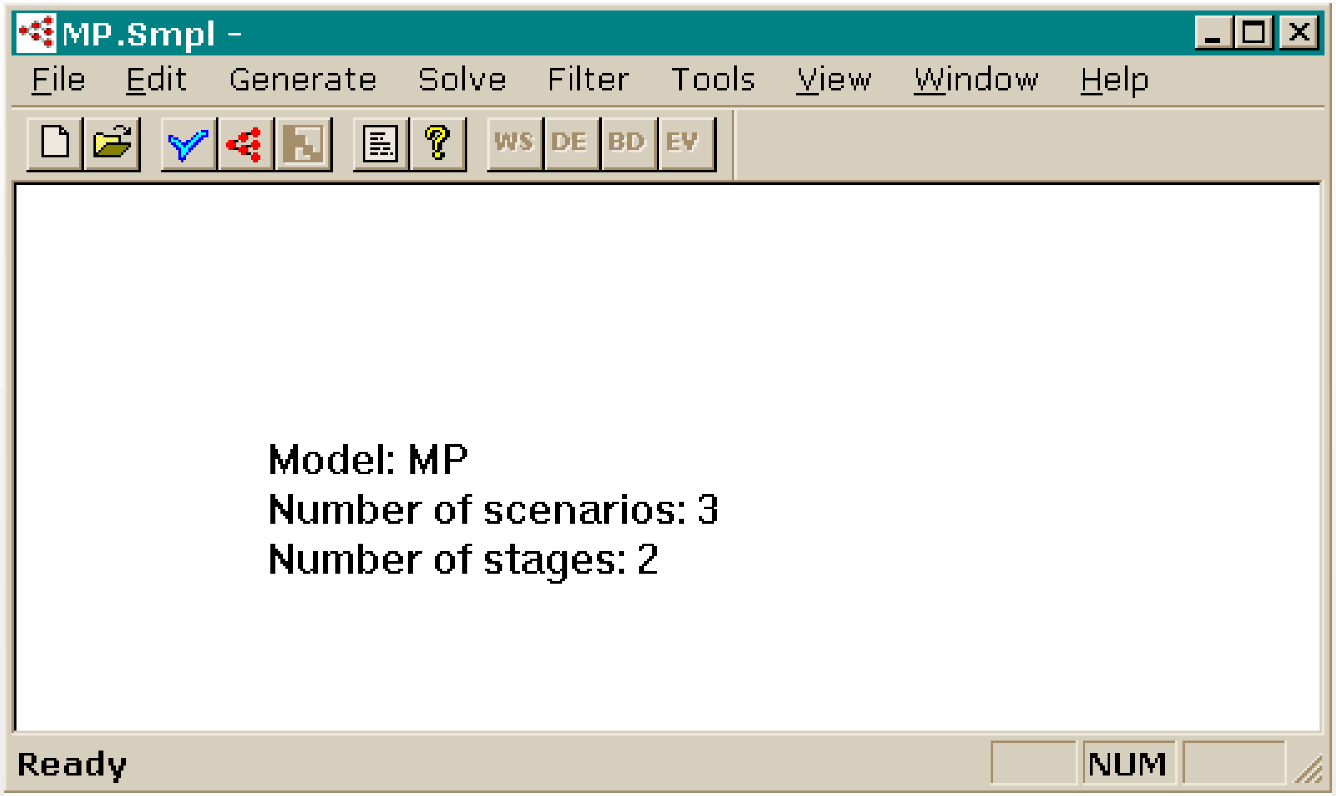
# Validation/Evaluation

- Assessment based on EVPI & VSS
- Simulation on out-of-sample scenarios

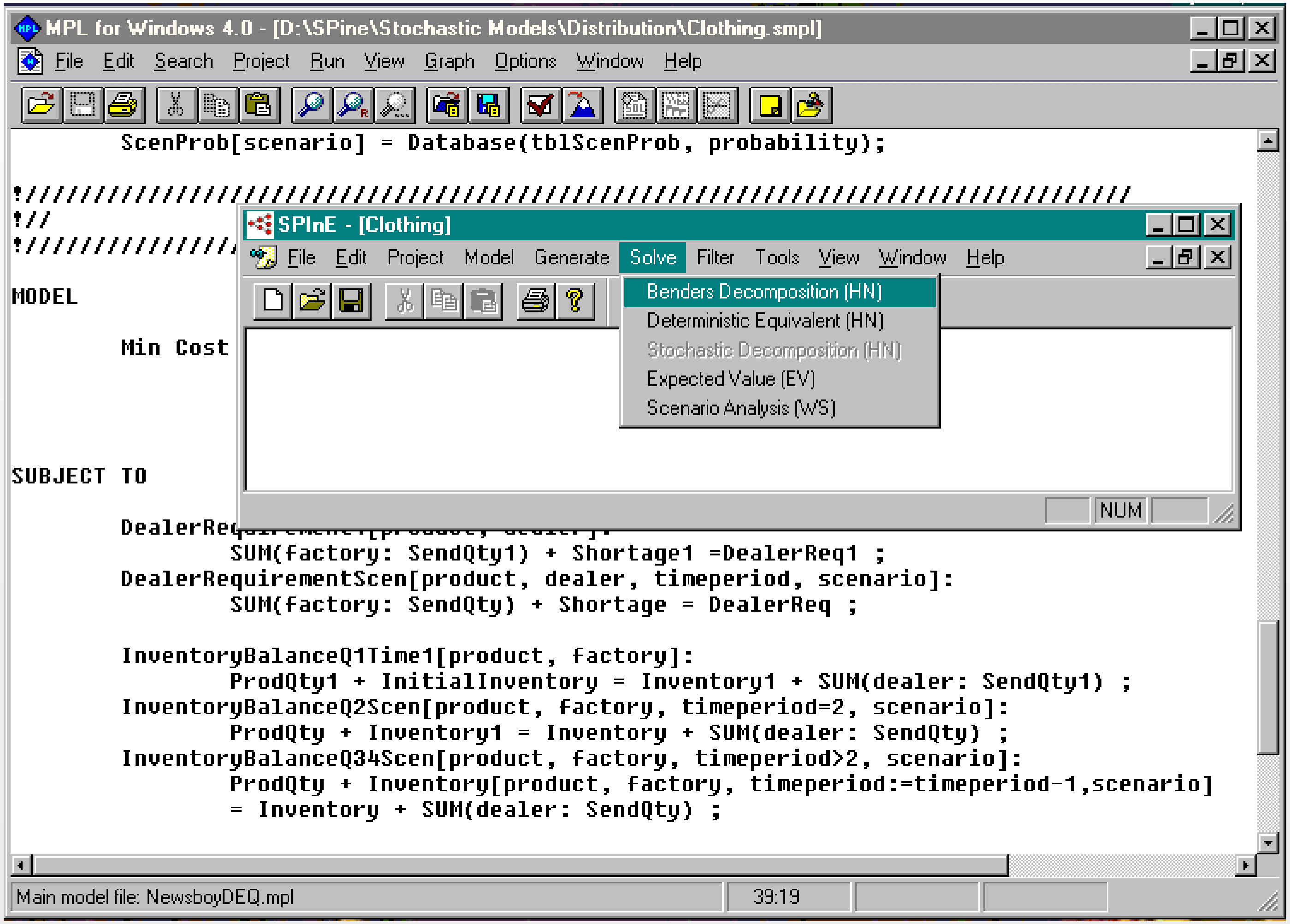
# Prototype user interface (1)



# Prototype user interface (2)



# Prototype user interface (3)



# References


Fantauzzi F.G. (1996): SPInE User's Manual, *Department of Mathematics and Statistics, Brunel University.*

Fourer R., Gay D.M. (1997): Stochastic Programming in the AMPL Modeling Language, *Session WE4-G-IN11 Modelling Support for Stochastic Programming, Intl. Symposium on Mathematical Programming, Lausanne, Aug 1997.*

Gassmann H.I., Ireland A.M. (1996): On the formulation of stochastic linear programs using algebraic modelling languages, *Annals of Operations Research 64, pp. 83-112.*

King A. (1994): SP/OSL Version 1.0 Stochastic Programming Interface Library: User' s Guide, *Research Report RC 19757 (87525) 9/26/94 Mathematics, T.J Watson Research Center, Yorktown Heights, NY.*

Messina E. , Mitra G. (1997): Modelling and analysis of multistage stochastic programming problems: A software environment, *European Journal Of Operational Research 101 2, pp. 343-359.*

MirHassani S.A. (1998): An investigation of Tools for Modelling and Solving Large Scale Linear Programming Problems under Uncertainty, *PhD Thesis, Department of Mathematics and Statistics, Brunel University.*

Valente P. (1999): Un ambiente integrato per la modellazione, risoluzione ed analisi di problemi di programmazione stocastica, *Tesi di Laurea in Scienze dell' Informazione, Universita' degli Studi, Milano Italy. To be submitted in April 1999.*