# RISK MEASURES AND MODELS FOR CHOICE UNDER UNCERTAINTY

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# STRUCTURE

1. The problem of choice under uncertainty. Example: the portfolio selection problem

2. Models for choice under uncertainty:
   - *mean-risk models*
   - *expected utility maximisation*
   - *stochastic dominance*

3. Compatibility between models for choice under uncertainty

4. Classification of risk measures

5. Axiomatic characterisations for risk measures

6. Overview of risk measures under the aspects of: compatibility with stochastic dominance, coherence

**The problem: deciding between 2 random variables when an outcome as large as possible is desirable.**

**Example: the portfolio selection problem.**

- An initial amount of money to invest, n assets, each asset i giving a return $R_i$ at the end of the investment period ($R_i$ random variable)

- If $x_i$ is the proportion of wealth invested in asset i, the portfolio return is $R_X = x_1 R_1 + \ldots + x_n R_n$ ($R_X$ random variable).

- For another portfolio resulting from choice $(y_1, \ldots, y_n)$, the portfolio return is the random variable $R_Y = y_1 R_1 + \ldots + y_n R_n$

- The problem: how do we choose between $R_X$ and $R_Y$?

# Models for choice under uncertainty

-Mean-risk models

-Expected utility maximisation

-Stochastic dominance

# Models for choice under uncertainty: Mean-risk models

•2 scalars attached to a r.v.: the ***mean*** (representing the expected outcome) and the ***risk*** (a scalar measure for the uncertainty of the outcomes).

•Let ρ be a risk measure, i.e. a function mapping random variables into real numbers.

Definition: In the mean-risk approach with risk measure given by ρ we say that r.v. ***$R_X$ is preferred to r.v. $R_Y$*** (or ***$R_X$ dominates $R_Y$***) if and only if: $E(R_X) \geq E(R_Y)$ and $\rho(R_X) \leq \rho(R_Y)$ with at least one strict inequality.

Notation: $R_X >_{m/\rho} R_Y$

•Definition: We say that the random variable $R_X$ is ***efficient*** (or ***nondominated***) if and only if there is no other r.v. $R_Y$ such that $R_Y$ dominates $R_X$.

•This means that for a given level of expected return $R_X$ has the lowest possible risk and for a given level of risk it has the highest return.

•The purpose of a mean risk model is to find efficient r.v. (portfolios).

# Models for choice under uncertainty: Expected utility maximisation

•Purpose: to quantify the desirability of a random variable by associating to it a real number - **its expected utility.**

•**A utility function** is a real valued function U defined on real numbers (representing possible wealth levels at the end of the investment period).

•Expected Utility Theory: extends a utility function defined on real numbers to a utility function defined on random variables.
An utility value E[U(X)] is assigned to each random variable X:

$$E[U(X)] = \int U(x)\,dF(x)$$

where F is the cumulative distribution function of X.

•Specifically, one compares 2 random variables by comparing their corresponding expected utility values; the larger value is preferred.

# Economic properties of utility functions

•<u>The problem</u>: what conditions a utility function U for money should satisfy such that the expected utility maximization criterion leads to rational decisions?

•Connection between **<u>risk attitudes</u>** that correspond to observed economic behaviour and **<u>the form of utility function</u>** to be used:

•**1.** ***Increasing wealth preference*** (more is preferred to less).
U ***increasing*** ($U'(w) \geq 0$ for all w with at least one strict inequality)

•**2.** ***Risk aversion*** (between a gamble and a sure thing with the same expected value, the sure thing is preferred)
U **increasing and *concave*** ($U'(w) \geq 0$, $U''(w) \leq 0$ for all w with at least one strict inequality, i.e. U has decreasing marginal utility)

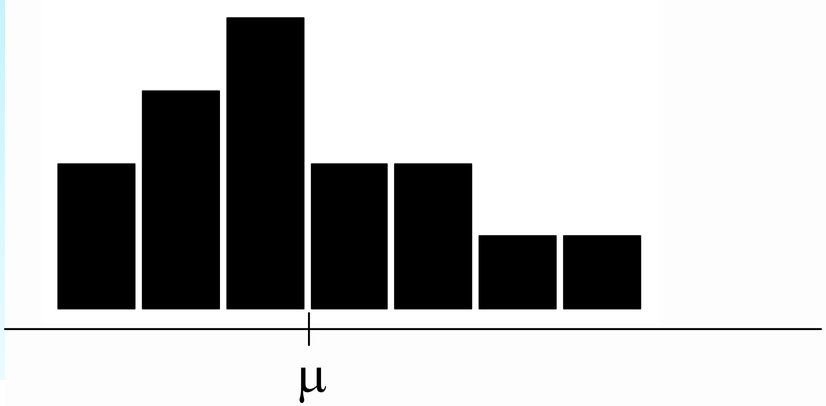
For a risk- averse investor: The expected utility of a risky investment is always less than the utility of its expected value: $E[U(X)] \leq U(E(X))$

# Economic properties of utility functions (cont.)

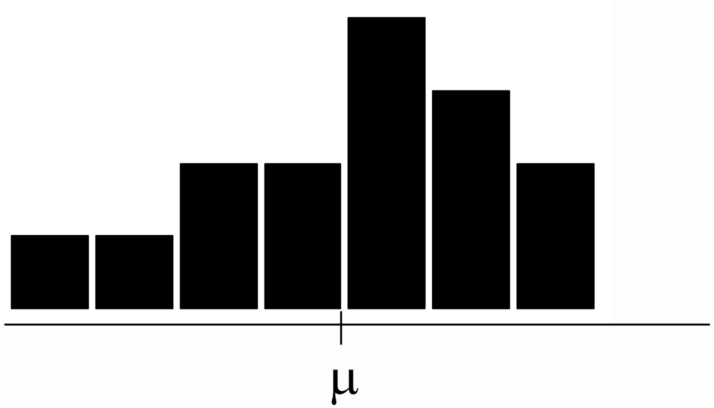
•3. ***Positive skewness preference (ruin aversion):*** unwillingness to accept small, almost certain gain in exchange for the remote possibility of ruin

U **increasing, concave and *U’ convex*** (***U’’’(w)≥0*, **U’(w)≥0, U’’(w)≤0 )

Positive skewness




Negative skewness



# Economic properties of utility functions (cont.)

•**4.** ***Decreasing absolute risk aversion***: increase the investments in risky assets as wealth increases (taking more risk when financially secure)
***A(W) decreasing*** (**A'(W)≤0**, where A(W) is the Arrow-Pratt absolute risk aversion coefficient)

$$A(W) = -\frac{U''(W)}{U'(W)}$$

A'(W)≤0 means: the bend in the graph of U is less stronger as wealth increases.

**<u>Conclusion</u>**: A utility function which is ***increasing, concave and its curvature reduces as wealth increases*** – tending to be linear at high level of wealth – corresponds to observed economic behaviour.

# Models for choice under uncertainty: Stochastic dominance (SD)

The purpose: to rank choices (random variables) under assumptions about general characteristics of utility functions (eliminates the need to explicitly specify a utility function).

Different types of SD correspond to different classes of utility functions defined previously.

SD is based on an axiomatic model of risk- averse preferences:

# Models for choice under uncertainty: Stochastic dominance

-1st order stochastic dominance (FSD): assumes an investor is rational (prefers more to less).
R.v. X is preferred to r.v. Y with respect to FSD ($X >_{FSD} Y$) if and only if
$E[U(X)] \geq E[U(Y)]$, for every increasing utility function U.
This means that X is preferred to Y by all rational investors.

-2nd order stochastic dominance (SSD): assumes the investor is rational and risk-averse
X is preferred to Y with respect to SSD ($X >_{SSD} Y$) if and only if
$E[U(X)] \geq E[U(Y)]$, for every U increasing and concave.
(X is preferred to Y by all rational investors who are risk averse)

# Models for choice under uncertainty: Stochastic dominance

-3rd order stochastic dominance (TSD): assumes an investor is rational, risk-averse and exhibits ruin-aversion

X is preferred to r.v. Y with respect to TSD ($X >_{TSD} Y$) if and only if:

$E(X) \geq E(Y)$ and

$E[U(X)] \geq E[U(Y)]$, for every U increasing, concave with U' convex.

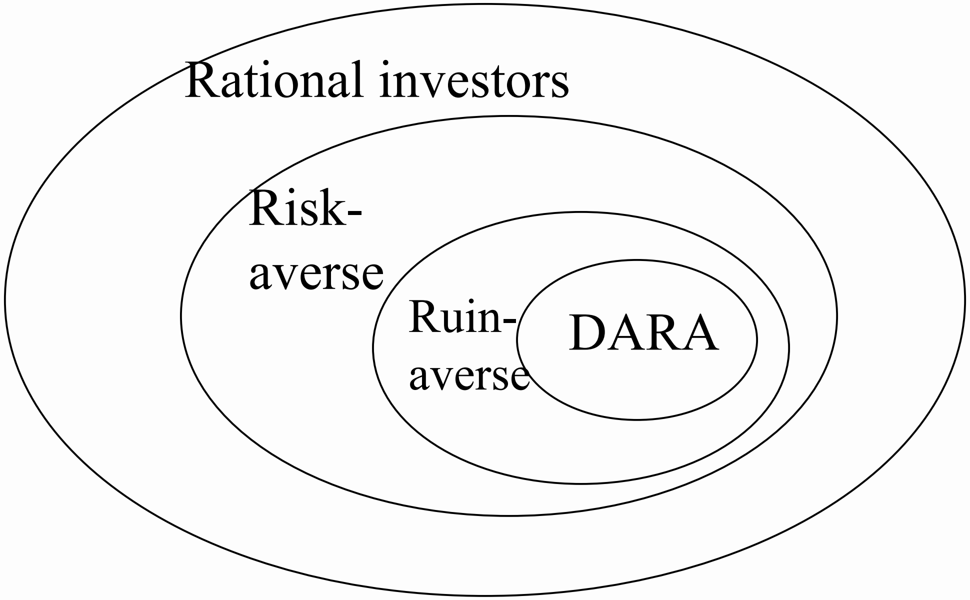
- Decreasing absolute risk aversion (DARA) stochastic dominance: assumes the investor is rational, risk-averse and exhibits decreasing absolute risk aversion

X is preferred to Y with respect to DARA ($X >_{DARA} Y$) if and only if:

$E(X) \geq E(Y)$ and

$E[U(X)] \geq E[U(Y)]$, for every U increasing, concave with $A'(W) \leq 0$.

# Models for choice under uncertainty: Stochastic dominance (SD)




## Relations between SD of different orders:

$$X >_{FSD} Y \Rightarrow X >_{SSD} Y \Rightarrow X >_{TSD} Y \Rightarrow X >_{DARA} Y$$

So: efficient solution with respect to DARA stochastic dominance ⇒ efficient w.r.t. TSD ⇒ efficient w.r.t. SSD ⇒ efficient w.r.t. FSD.

# Compatibility between models for choice under uncertainty

•Definition: We say that the mean risk model with the risk measure given by ρ is ***consistent* with a SD relation** if and only if: $X >_{SD} Y$ implies $X >_{m/\rho} Y$ for every random variables X, Y.

It means that: **efficient portfolios (r.v.) in the mean-risk model are also nondominated with respect to stochastic dominance relations**, so the mean-risk model leads to rational decisions (corresponding to observed economic behaviour).

•Definition: We say that the mean risk model with the risk measure given by ρ is ***congruent* with a SD relation** if: $X >_{SD} Y \Leftrightarrow X >_{m/\rho} Y$ for every random variables X, Y.

This means that the 2 models provide the same ranking of choices.

# Classification of risk measures

**I. Risk measures of the first kind**: measure the magnitude of deviations from a target.

When the target is the mean of the random variable, they are called **deviation measures**.

-two-sided - penalise negative as well as positive deviations from a target: **variance, MAD.**
-one-sided (downside)- penalise only the negative deviations from a target: **lower partial moments**.

They can have only positive values.

# Classification of risk measures

**II. Risk measures of the second kind**: measure the overall seriousness of possible **losses**.

Used only in financial context

Can have both positive and negative values

Can be regarded as:
-necessary capital (to be added to a position in order to make it riskless) in case the value of the risk measure is positive;
-necessary premium (to be withdrawn from a position without endangering safety) in case the risk measure is negative.

Examples: Value-at-Risk (VaR), Conditional Value-at-Risk (CVaR).

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# Axiomatic characterisations of risk measures of second kind. Coherence

1. The **coherence** is defined by means of several axioms which are desirable properties for a risk measure ρ **of the second kind**:

•***risk-free condition (translation invariance):*** adding a sure initial amount A to the initial portfolio and investing it in the risk free asset, it decreases the risk measure by A.
$\rho(X+A)= \rho(X)-A$, ∀ X r.v., A real no (for the left-hand side: A is the corresponding degenerate r.v.)

•***positive homogeneity*****:** position size directly influences risk
$\rho(aX)= a\rho(X)$, ∀ X r.v., a positive real number.

CARISMA

# Axiomatic characterisations of risk measures of second kind. Coherence

•***subadditivity***: a merger does not create extra risk

$\rho(X+Y) \leq \rho(X) + \rho(Y)$, $\forall$ X, Y r.v.

•***Monotonicity***: if all possible outcomes for one choice are less than the outcomes for another choice, then the first should have a greater risk.

$X \leq Y$ ***a.s.*** $\Rightarrow \rho(X) \geq \rho(Y)$.

# Axiomatic characterisations of deviation measures

•**positive homogeneity** (position size directly influences risk)
$D(aX) = aD(X)$, $\forall$ X r.v., a real positive no.

•**subadditivity** (a merger does not create extra risk)
$D(X+Y) \leq D(X) + D(Y)$, $\forall$ X, Y r.v.

•**Translation invariance** (adding a constant to a r.v. we will obtain a new r.v. with the same deviation around the mean )
$D(X+A) = D(X)$, $\forall$X r.v., $\forall$A constant r.v.

•**Sign condition** (any nondegenerate r.v. has a strictly positive deviation from its mean)
$D(X)>0$, $\forall$ X nonconstant r.v., $D(X)=0$, $\forall$ X constant r.v

## Correspondence between risk measures of the first kind and risk measures of the second kind

•A one_to_one correspondence is set between deviation measures satisfying the above 4 conditions and risk measures of the second kind satisfying a slightly modified version of coherence under the relations:

$D(X)=\rho(X-EX)$

$\rho(X)=D(X)-EX.$

**•A deviation measure D is said to be coherent if the corresponding risk measure of the second kind ($\rho(X)=D(X)-EX$) is coherent.**

# Overview of risk measures. Deviation measures

•**Standard deviation:**

$$\sigma(X) = \{E[(X - E(X))^2]\}^{\frac{1}{2}}$$

•**Mean Absolute Deviation (MAD):**

$$MAD(X) = E[/ X - E(X) /]$$

•**Lower partial moments of order α around mean:**

$$LPM_{\alpha}(EX, X) = [E\{[\max(0, EX - X)]^{\alpha}\}]^{\frac{1}{\alpha}}$$

***All of them satisfy positive homogeneity, subadditivity, translation invariance and sign condition.***

<u>The questions</u>:

1. Compatibility with SD
2. Coherence

# Overview of risk measures. Deviation measures

| | Coherency | Compatibility with SD |
|---|---|---|
| Standard deviation | **No** (monotonicity not satisfied) | -in general, no compatibility<br>-for special classes of distributions, **congruence** with SSD |
| MAD | **No** (monotonicity not satisfied) | -in general, no compatibility<br>-for special classes of distributions, **congruence** with SSD |
| LPM of order $\alpha$ ($\alpha \geq 1$) around mean | **Yes** | -in general, no compatibility |

# Overview of risk measures: Lower partial moments around fixed target

They are **downside risk measures of first kind** (measure the magnitude of negative deviations from a fixed target return $\tau$)

***The lower partial moment of order $\alpha$ ($\alpha \geq 0$) around $\tau$ associated to a r.v. $X$*** is

$$LPM_{\alpha}(\tau, X) = E\{[\max(0, \tau - X)]^{\alpha}\}$$

**($\alpha$, $\tau$) model** is the mean-risk model in which the risk measure is given by LPM of order $\alpha$ around $\tau$.

**-$\alpha$ characterizes the investor's attitude towards risk.**
$\alpha \in (0,1)$ is appropriate when the investor is risk – seeking
$\alpha=1$ suits a risk-neutral investor
$\alpha>1$ is appropriate when the investor is risk – averse

# Overview of risk measures: Lower partial moments around fixed target

## Compatibility with stochastic dominance:

- **<u>for $\alpha \geq 0$</u>: <u>consistency with FSD</u>** (the efficient solutions of the $(\alpha,\tau)$ model are optimal for every rational investor).

- **<u>for $\alpha \geq 1$</u>: <u>consistency with SSD</u>** (the efficient the efficient solutions of the $(\alpha,\tau)$ model are optimal for every rational and risk-averse investor)

- **<u>for $\alpha \geq 2$</u>: <u>consistency with TSD</u>** (the efficient the efficient solutions of the $(\alpha,\tau)$ model are optimal for every rational, risk-averse and ruin averse investor)

# Risk measures of the second kind: VaR

Let the losses associated with a choice $x=(x_1,\ldots,x_n)$ be given by a random variable $L_{X.}$(usually, $L_X=-R_X$).

**VaR at a confidence level $\beta\in(0,1)$ associated with choice x (or with random variable $R_X$) is the maximum loss at the end of the investment period within this confidence level.**

Usually, $\beta=0.95$ or $\beta=0.99$.

Let F be the cumulative distribution function of the r.v. $L_{X.}$

$$F(\alpha) = \Pr ob(L_X \le \alpha)$$

**$VaR_\beta(x)= VaR_\beta(R_X)=VaR_\beta(L_X)=\min\{\alpha\in R$ such that $F(\alpha)\geq\beta\}$**

# Risk measures of the second kind: CVaR

**CVaR at a confidence level $\beta \in (0,1)$ associated with choice x (or with random variable $R_X$) is the "average loss in the worst $(1-\beta)\%$ cases".**

$CVaR_\beta(x) = CVaR_\beta(R_X) = CVaR_\beta(L_X)$ = the mean of the $\beta$-tail distribution of $L_X$.

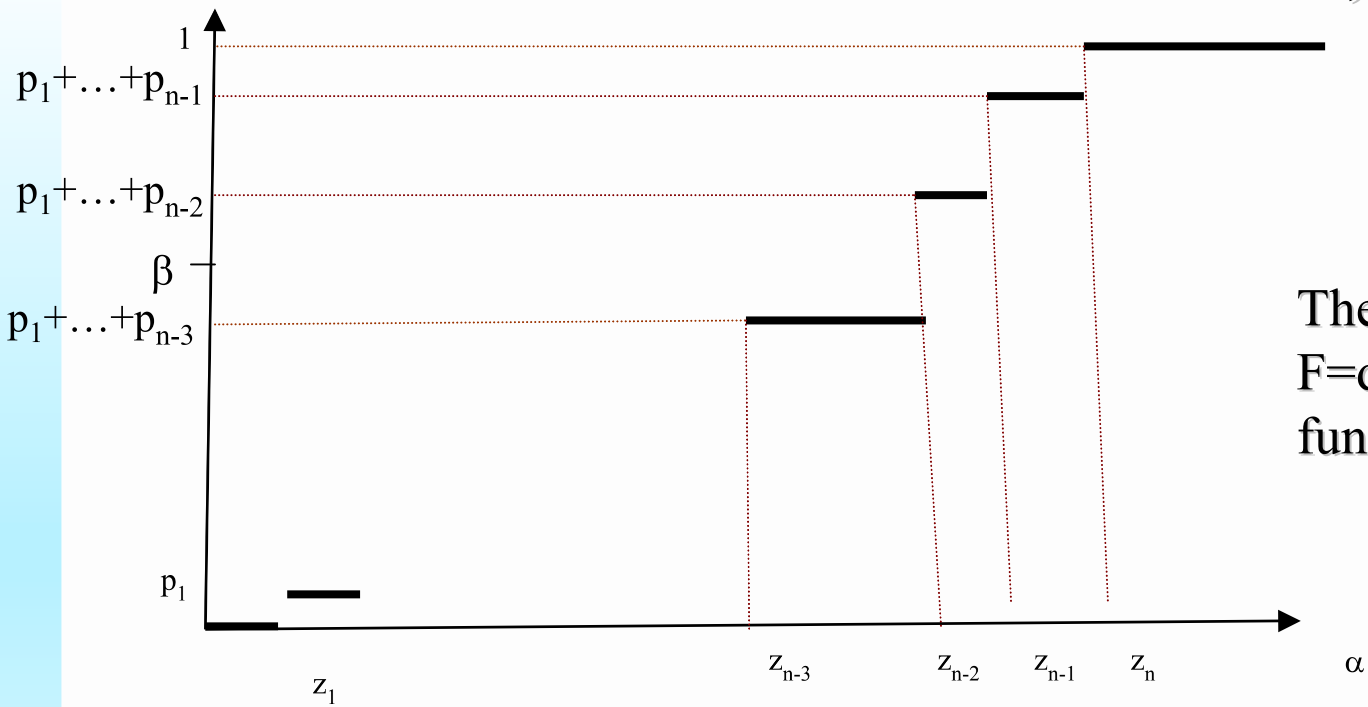
The $\beta$-tail distribution is obtained by taking the upper $(1-\beta)$ part of the distribution of $L_X$ (corresponding to extreme losses) and rescaling it to fit [0,1].

The distribution function of the $\beta$-tail is $F_\beta$:

$$F_\beta(\alpha) = 0 \quad \text{for } \alpha < VaR_\beta(x)$$

$$F_\beta(\alpha) = \frac{F(\alpha) - \beta}{1 - \beta} \quad \text{for } \alpha \geq VaR_\beta(x)$$

# Risk measures of the second kind: VaR, CVaR




The graph of F=distribution function of $L_X$.

Let $L_X$ be finitely distributed $L_X$: $\begin{matrix} z_1 & z_2 & \dots & z_{n-2} & z_{n-1} & z_n \\ p_1 & p_2 & \dots & p_{n-2} & p_{n-1} & p_n \end{matrix}$

If $\beta \in (p_1+\dots+p_{n-3},\ p_1+\dots+p_{n-2})$ then: $VaR_\beta(x) = z_{n-2}$

The β-tail: $\begin{matrix} z_{n-2} & z_{n-1} & z_n \\ \dfrac{p_1+\dots+p_{n-2}-\beta}{1-\beta} & \dfrac{p_{n-1}}{1-\beta} & \dfrac{p_n}{1-\beta} \end{matrix}$

$$CVaR_\beta(x) = \frac{1}{1-\beta}[z_{n-2}(p_1+\dots+p_{n-2}-\beta) + z_{n-1}p_{n-1} + z_n p_n]$$

# Risk measures of the second kind: VaR, CVaR

**In case the loss distribution $L_X$ is continuous, $CVaR_\beta(x)$ is the conditional expectation of losses above $VaR_\beta(x)$:**
**$CVaR_\beta(x)=E(L_X/ L_X>VaR_\beta(x))= E(L_X/ L_X\geq VaR_\beta(x))$**

**For discrete distributions (as in scenario models), the 3 values above are different.**

# Risk measures of the second kind: VaR

1. **Compatibility with stochastic dominance:**

-$\forall \beta \in (0,1)$, the mean- ${}_{\beta}$VaR model is consistent with FSD

- for special classes distributions (the normal one in particular) and if the random variables considered have the same mean, VaR is consistent with SSD.

**2. Coherence**

-in general, VaR is not coherent (lacks subadditivity; satisfies positive homogeneity, monotonicity and risk-free condition)

- for special classes distributions (the normal one in particular), VaR is coherent.

# Risk measures of the second kind: CVaR

1. **Compatibility with stochastic dominance:**

   -$\forall \beta \in (0,1)$, the mean- $_{\beta}$CVaR model is consistent with SSD

**2. Coherence**

CVaR is coherent.

# Conclusions

**Consistency of risk measures with stochastic dominance** is important in portfolio optimisation- it ensures that the efficient portfolios of the mean-risk model are also efficient by stochastic dominance rules.

**Coherence of risk measures** is important in insurance – asset/ liability management (in setting of the minimum of required capital for a position)