Computational Aspects of Alternative Portfolio Selection Models in the Presence of Discrete Asset Choice Constraints

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University Finance Seminar, University of Cambridge
12 May 2000

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Outline

- Mean-Variance Model
- Discrete Constraint Efficient Frontier (DCEF)
  - discrete constraints, constructing the DCEF
- Mathematical Representation
  - Markowitz’s QP-model with discrete constraints
  - invisible sections of the DCEF
- Modeling and Solution Paradigm
  - ‘integer-restart’ and ’reoptimisation’ heuristic
- Computational Results
  - sample DCEF’s
  - comparision of the heuristics for 5 benchmark datasets
  - portfolio rebalancing problem

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Mean-Variance Model

- Markowitz (1952,1959)
- alternative formulations

QP1

$$\begin{aligned} Min \quad & Z_{QP1} = \sum_{i=1}^{N}\sum_{j=1}^{N} x_i\, x_j \sigma_{ij} \\ s.t. \quad & \sum_{i=1}^{N} x_i \mu_i = \rho \\ & \sum_{i=1}^{N} x_i = 1 \\ & x_i \geq 0, \quad i = 1,\ldots,N \end{aligned}$$

QP2 (Arrow-Pratt absolute risk aversion index)

$$\begin{aligned} Max \quad & Z_{QP2}^{R_A} = \frac{R_A}{2}\sum_{i=1}^{N} x_i \mu_i - \sum_{i=1}^{N}\sum_{j=1}^{N} x_i\, x_j \sigma_{ij} \\ s.t. \quad & \sum_{i=1}^{N} x_i = 1 \\ & x_i \geq 0, \quad i = 1,\ldots,N \\ & R_A \geq 0 \end{aligned}$$

$i, j = 1,\ldots,N$: denotes the different risky assets

μ_i : expected return of asset i

σ_{ij} : covariance between asset i and asset j

ρ : desired level of return

x_i : the fraction of portfolio value invested in asset i

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Mean-Variance Model

- Arrow Pratt absolute Risk Aversion Index

$$R_A = \frac{u''(w)}{u'(w)}$$

where

w is the portfolio wealth

u a Von Neumann-Morgenstern utility function with first and second derivatives

- Portfolios with similar Absolute Risk Aversion index yield in similarportfolios (weight-vector) regardless of functional form and parameters of the utility function (Kallberg and Ziemba 1983)

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Mean-Variance Model

QP2 (Lambda-formulation)

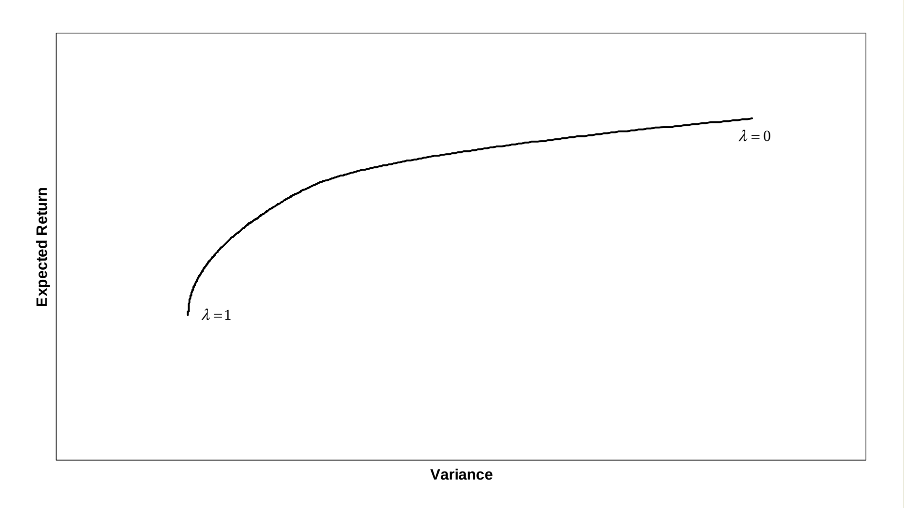
$$Min \qquad Z_{QP2} = \lambda \sum_{i=1}^{N} \sum_{j=1}^{N} x_i \, x_j \sigma_{ij} - (1-\lambda) \sum_{i=1}^{N} x_i \mu_i$$

$$s.t. \qquad \sum_{i=1}^{N} x_i = 1$$

$$x_i \geq 0, \quad i = 1, \ldots, N$$

$$0 \leq \lambda \leq 1$$

Efficient Frontier



Discrete Constraint Efficient Frontier (DCEF) (1)

- Buy-in thresholds
  - min. level below an asset is not traded
  - eliminates unrealistically small trades
- Cardinality Constraints
  - controls the number of stocks in a portfolio
  - monitoring and control issues (management effort)
- Roundlots
  - trades only in multiples of ‘discrete’ numbers of assets possible

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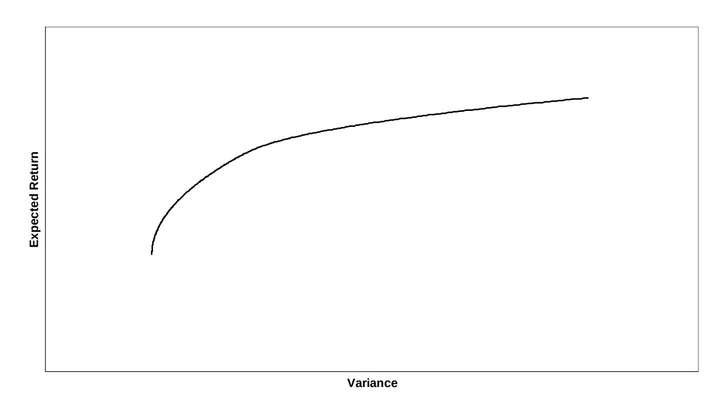
Discrete Constraint Efficient Frontier (DCEF) (2)

Example:

- 4 stock universe, EF



- introduce cardinality constraint, i.e. build a portfolios containing 2 stocks only

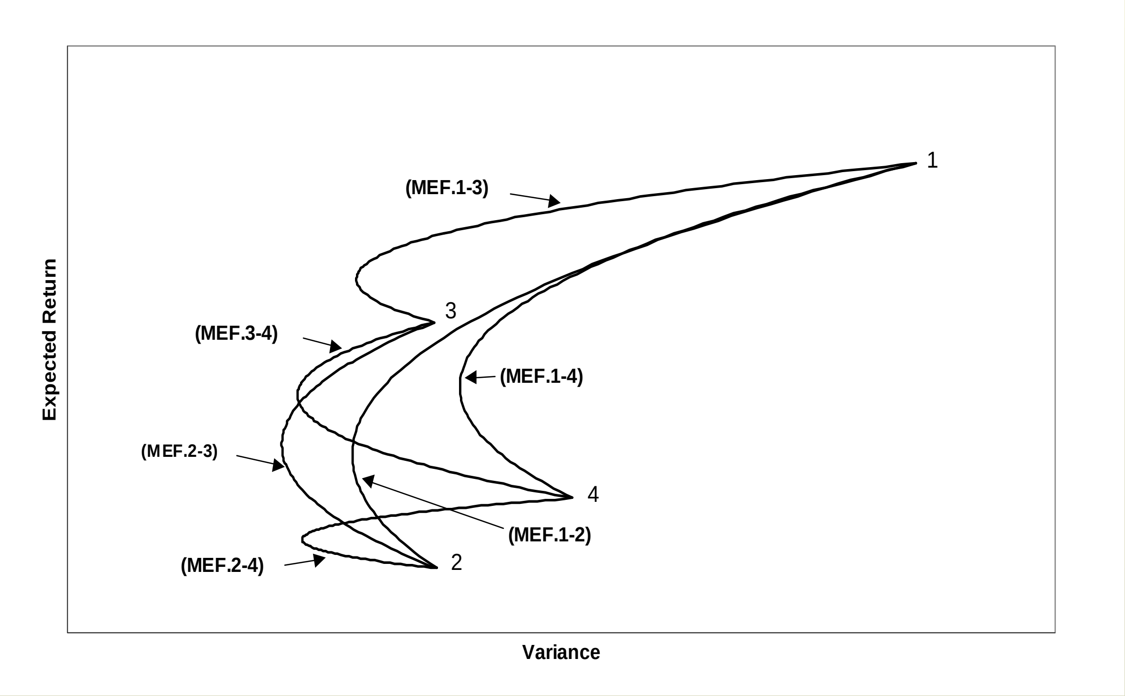
Effect:

Discontinuities in the Efficient Frontier

Discrete Constraint Efficient Frontier (DCEF) (3)

Why discontinuities?

- take investment opportunity set
- delete all inefficient portfolios (dominated points)



Mathematical Representation (1)

Extending QP1 with discrete constraints

- Buy-in thresholds

$$Min \quad Z_{BUY-IN} = \sum_{i=1}^{N}\sum_{j=1}^{N} x_i\, x_j \sigma_{ij}$$

$$s.t. \quad \sum_{i=1}^{N} x_i \mu_i = \rho$$

$$\sum_{i=1}^{N} x_i = 1$$

$$l_i \delta_i \le x_i \le u_i \delta_i, \quad i = 1,\ldots, N$$

$$\delta_i \in \{0,1\}, \quad i = 1,\ldots, N$$

l_i, u_i : lower and upper bound on the stock weight

δ_i : binary variable

- Cardinality Constraints

$$\sum_{i=1}^{N} \delta_i = k$$

k: number of assets

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Mathematical Representation (2)

- Transaction Roundlots
  - integer number of blocks y_i
  - a lot can be illustratively expressed as fraction f_i of the portfolio wealth
  - re-express x_i as $x_i = y_i f_i, \quad i = 1,\ldots,N$

$$Min \quad Z_{LOT} = \sum_{i=1}^{N}\sum_{j=1}^{N} y_i f_i y_j f_j \sigma_{ij} + \gamma(\varepsilon^- + \varepsilon^+)$$

$$s.t. \quad \sum_{i=1}^{N} y_i f_i \mu_i = \rho$$

$$\sum_{i=1}^{N} y_i f_i + \varepsilon^- - \varepsilon^+ = 1$$

$$l_i \le y_i f \le u_i, \quad i = 1,\ldots,N$$

$$y_i \quad \text{integer}, \quad i = 1,\ldots,N$$

$$\varepsilon^-, \varepsilon^+ \ge 0$$

$\varepsilon^-, \varepsilon^+$: undershoot, overshoot variable

γ : penalty

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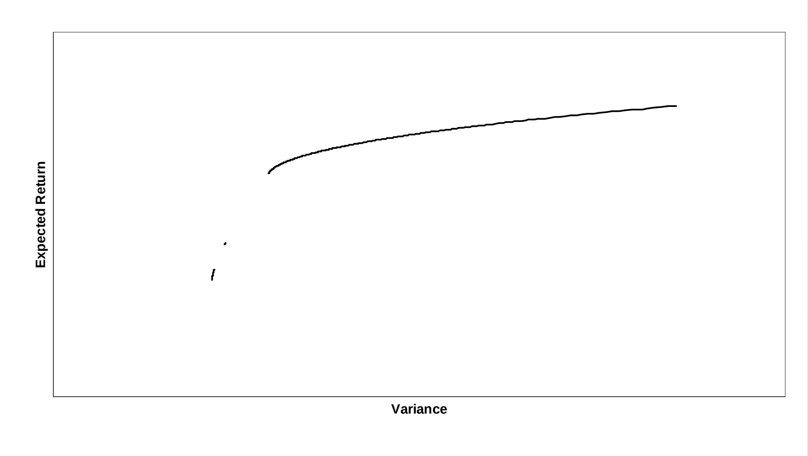
Invisible Sections of the DCEF (1)

- 4 stock example, cardinality constraint k=2
  - QP1 underlying

- Add discrete restrictions to QP2 ("Lambda formulation")
  - QP2 underlying



<u>Effect:</u>

reduced efficient

frontier (missing parts)

Modeling, Solution Paradigm (1)

- Tracing the efficient frontier requires the solution to quadratic mixed integer program for every point on the curve (QMIP) $\Rightarrow$ very expensive and not practicable for reasonable size problems
- Modern heuristics (GA, SA, TS), e.g. Chang et al. (1999)
- Heuristics developed based on Branch & Bound search with SSX for sub-problem solution
  - Integer restart heuristic
  - Reoptimisation heuristic

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Modeling, Solution Paradigm (2)

- **Integer Restart Heuristic**
  - solving say 500 QMIPS to optimality is unlikely, therefore it is likely to loose the "pareto efficient" property of the DCEF
  - Method starts from the highest return level and continues computing solutions for lower levels of return
  - uses the previous integer solution as the 'first feasible and upper bounding QP' value for computing the next point
  - restricted B&B search (stopped after fixed number of nodes)
  - within the band of sub-optimality, we retain the 'efficient' property of the frontier, i.e. risk and return increases or decreases jointly

Modeling, Solution Paradigm (3)

- **Reoptimisation Heuristic**
  - reflects common practice (in absence of a QMIP solver)
  - 2 QP models solved for each QMIP CARD problem

  - QP1 is solved ignoring size and threshold constraints
  - if at least k (cardinality) stocks are in solution
    - QP1 is solved again using a reduced universe of k stocks (with the largest weights in solution)
    - lower bound (buy-in threshold) is explicitly imposed in this run

  - Portfolio contains k stocks (satisfying buy-in threshold restriction)

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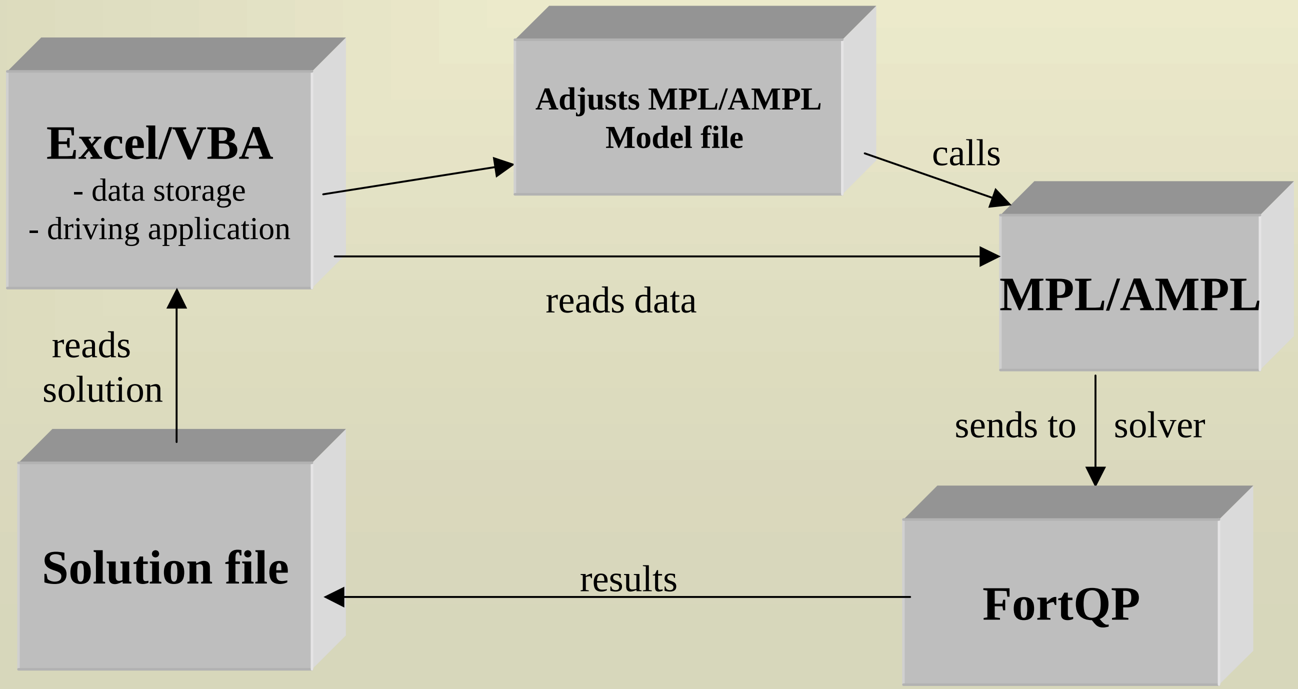
Modeling, Solution Paradigm (4)

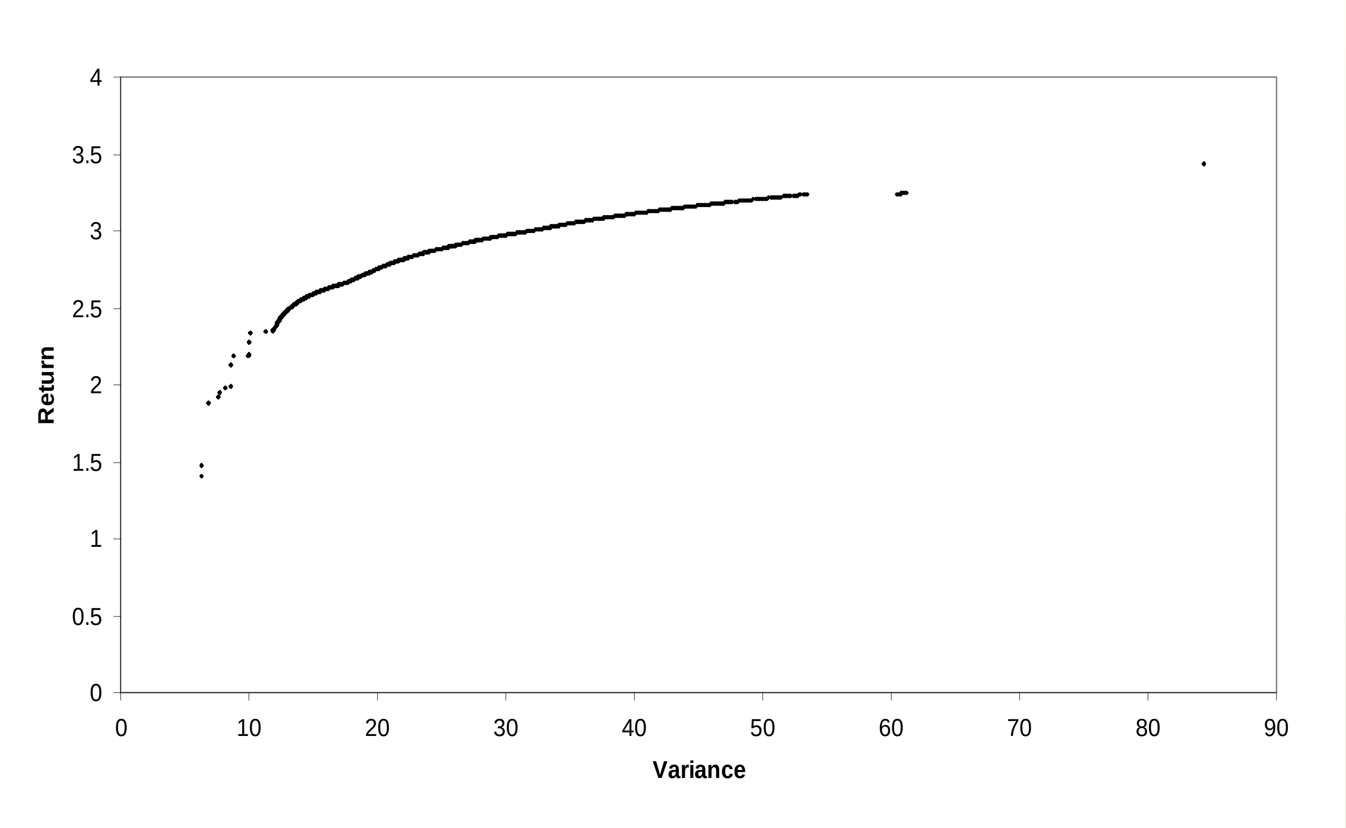
- Software tools:
  - Solver System (FortQP)
  - Mathematical Programming Modeling language (MPL/ AMPL)
  - EXCEL/VBA - application

Data - Modelling - Solver Architecture



Sample Frontiers (1)

- 30 stocks from FTSE 100
  - Buy-in threshold of 20%



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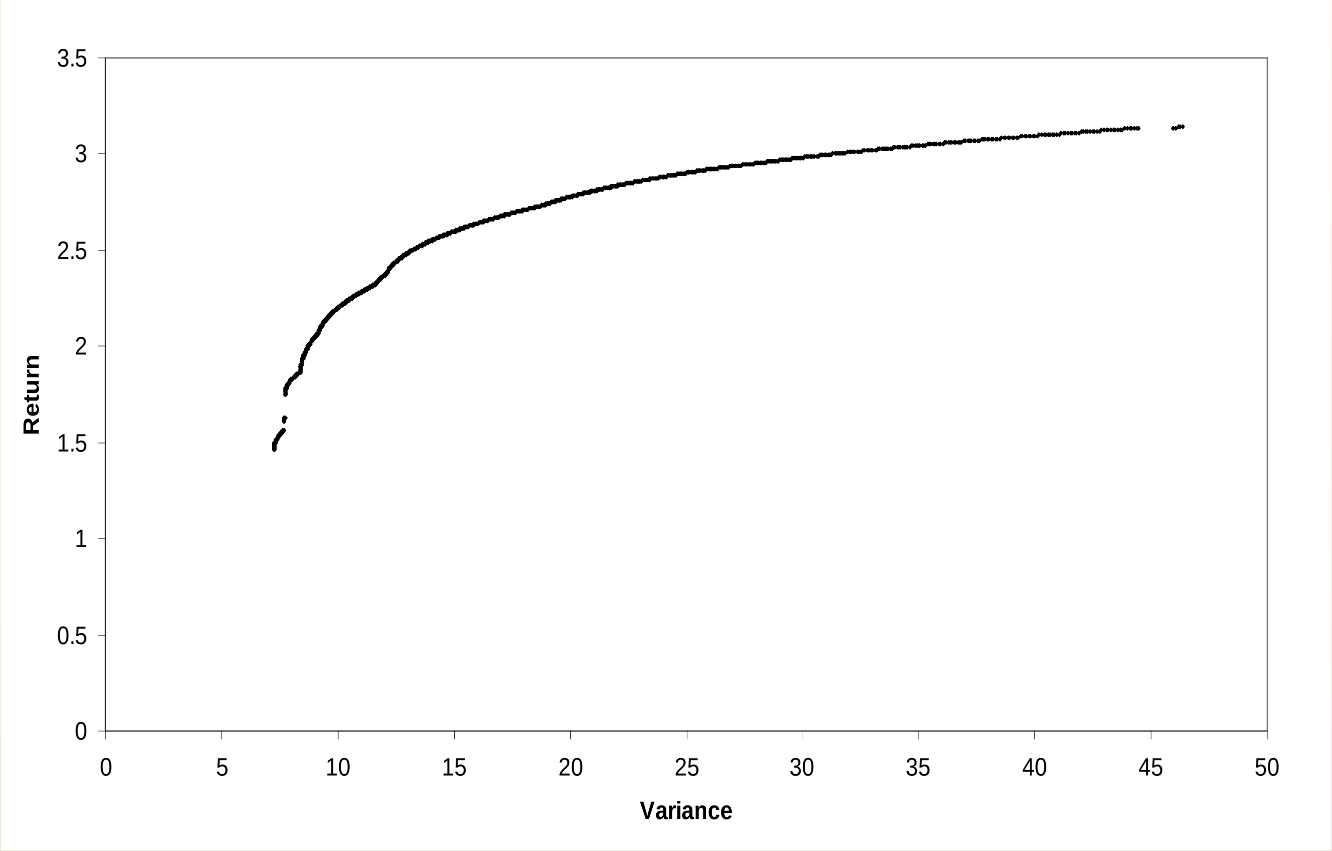
Sample Frontiers (2)

- Cardinality Constraint k=4, 10% threshold



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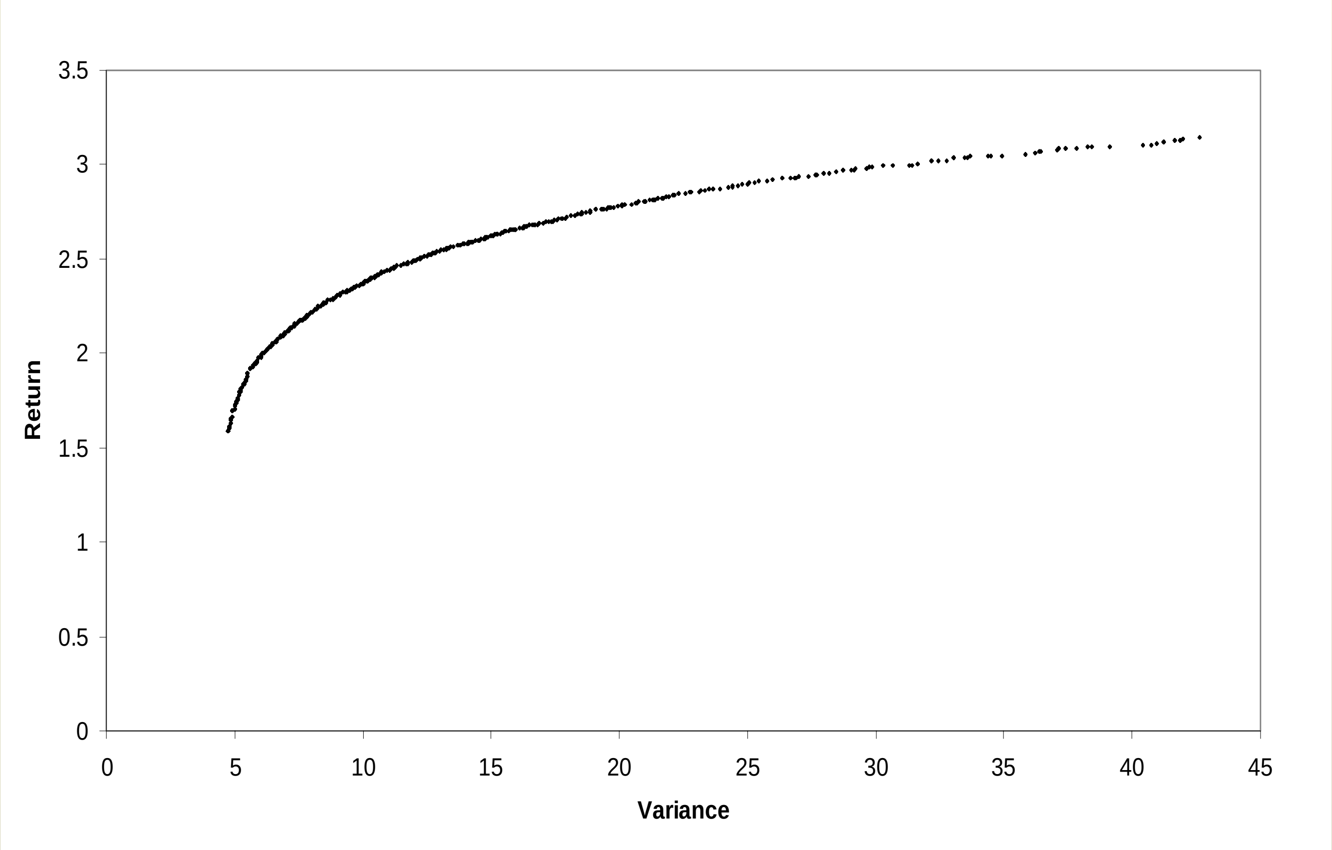
Sample Frontiers (3)

- Roundlot model



Computational Study (1)

- **Datasets**: taken from 5 markets
  - FTSE 100, Nikkei, Dax, S&P, Hang Seng
- **Model**: Cardinality Constraint
  - cardinality restriction: $k=10$
  - lower bound: $l_i=0.01$
- **Results** for both heuristics and comparison to modern heuristic approaches (Chang et al. (1999) are obtained
- **Metric** of comparison
  - integer optimal DCEF not available, we applied the metric proposed by Chang et al. (1999) and measure the distance to the Efficient Frontier (QP-optimal)

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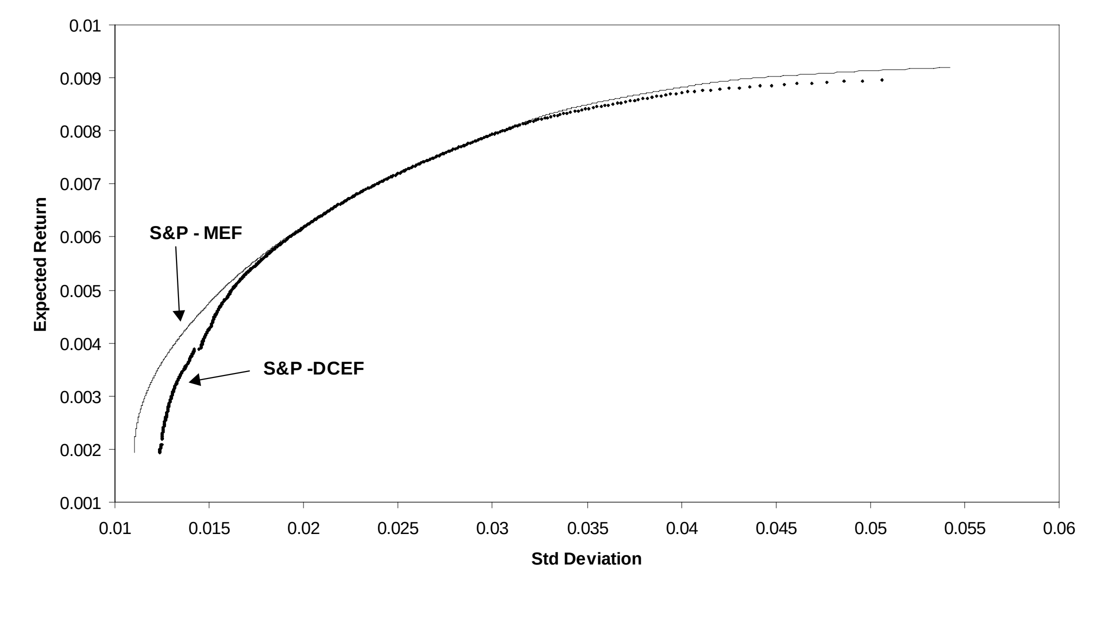
Computational Study (2)

- Integer Restart Heuristic

* Pentium 500, 128 MB RAM

Index	No. of Stocks	Total no. of DCEF pts	No. of integer optimal pts	Solution time *	Mean Error	Median Error
Hang Seng	31	500	492	57.55	0.01415	0.00997
DAX	85	500	228	8405.33	0.01399	0.01159
FTSE	89	500	244	10978.12	0.01141	0.00860
S & P	98	500	192	15831.97	0.01586	0.01325
Nikkei	225	500	486	18345.56	0.00618	0.00252



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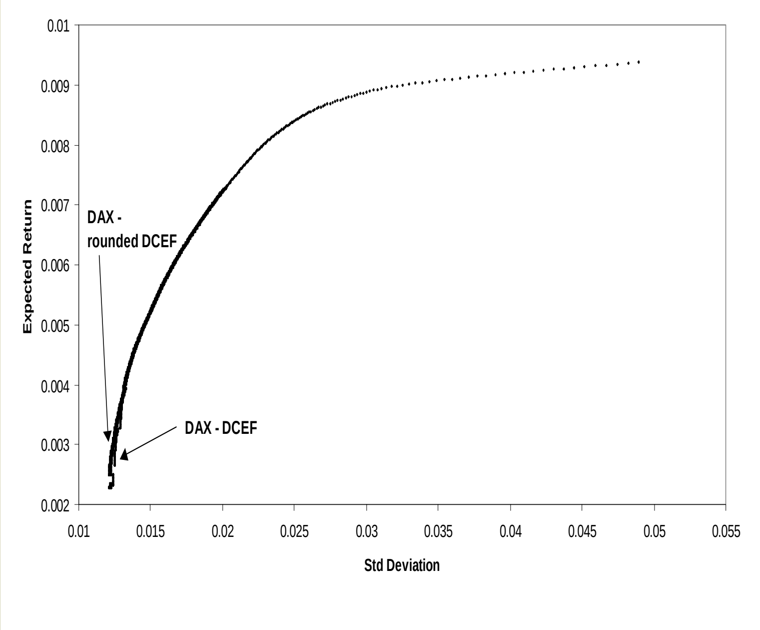
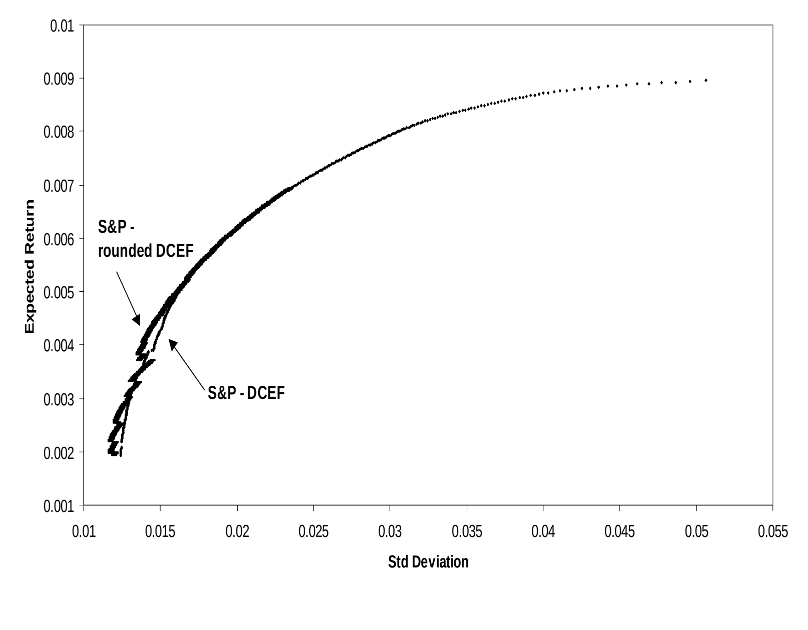
Computational Study (3)

Rounding Heuristic

* Pentium 500, 128 MB RAM

Index	No. of Stocks	Total no. of MEF pts	No. of discrete pts	No. of DCEF pts	Solution time *	Mean Error	Median Error
Hang Seng	31	500	104	103	10	0.00021	0.00051
DAX	85	500	356	349	37.53	0.01444	0.01155
FTSE	89	500	375	355	36.18	0.01014	0.00715
S & P	98	500	356	278	44.93	0.01652	0.01356
Nikkei	225	500	376	374	280.92	0.00316	0.00151





Computational Study (4)

- Comparison to modern heuristic approaches

Index	No. of Stocks	Solution Method	No. of efficient points	Mean Error	Median Error
Hang Seng	31	Integer restart heuristic	500	0.01415	0.00997
			3000	0.00826	0.00628
		Rounding heuristic	103	0.00021	0.00051
		GA heuristic *	1317	0.94570	1.18190
		TS heuristic *	1268	0.99080	1.19920
		SA heuristic *	1003	0.98920	1.20820
		pooled (GA, TS, SA) *	2491	0.93320	1.18990
DAX	85	Integer restart heuristic	500	0.01399	0.01159
		Rounding heuristic	349	0.01444	0.01155
		GA heuristic *	1270	1.95150	2.12620
		TS heuristic *	1467	3.06350	2.53830
		SA heuristic *	1135	2.42990	2.46750
		pooled (GA, TS, SA) *	2703	2.19270	2.46260

No direct comparison possible as results for modern heuristics are not available to the same detail !

*** Chang et al. (1999)**

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Computational Study (5)

- Comparison to modern heuristic approaches

FTSE	89	Integer restart heuristic	500	0.01141	0.00860
		Rounding heuristic	355	0.01014	0.00715
		GA heuristic *	1482	0.87840	0.59600
		TS heuristic *	1301	1.39080	0.71370
		SA heuristic *	1183	1.13410	0.63610
		pooled (GA, TS, SA) *	2538	0.77900	0.59380
S & P	98	Integer restart heuristic	500	0.01586	0.01325
		Rounding heuristic	278	0.01652	0.01356
		GA heuristic *	1560	1.71570	1.14470
		TS heuristic *	1587	3.16780	1.14870
		SA heuristic *	1284	2.69700	1.12880
		pooled (GA, TS, SA) *	2759	1.31060	1.06860
Nikkei	225	Integer restart heuristic	500	0.00618	0.00252
		Rounding heuristic	374	0.00316	0.00151
		GA heuristic *	1823	0.6431	0.6062
		TS heuristic *	1701	0.8981	0.5914
		SA heuristic *	1655	0.637	0.6292
		pooled (GA, TS, SA) *	3648	0.569	0.5844

No direct comparison possible as results for modern heuristics are not available to the same detail !

*** Chang et al. (1999)**

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Computational Study (6)

- General conclusions:
  - Computing the entire DCEF to optimality is computationally challenging
  - ***Integer restart heuristic*** computes a reasonable number of optimal or near optimal points within a restricted B&B search
  - ***Reoptimisation heuristic*** is computationally very efficient
    - entire frontier may not be computed
    - does not guarantee efficient points
  - Both methods outperform modern heuristic approaches (reported average errors are about 1%)
  - Real application, only a segment of the frontier may be of interest, Integer restart approach can be used to **"zoom in"** and compute few alternative portfolios exact or at least more accurate

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Computational Study (7)

- Portfolio Rebalancing Problem
  - cardinality constraints (limit on the number of trades)
  - additional variables and constraints:
  - balance constraint $x_i = n_i + b_i - s_i$

    x_i, b_i, s_i hold, buy, sell - variables for asset i

    n_i initial holding in asset i
  - binary variables δ_i^s, δ_i^b - indicator for either buying or selling
  - restriction that asset i can't be sold and bought at the same time

$$\delta_i^s + \delta_i^b \leq 1$$

  - buy-in threshold and upper bounds on buy/sell variables

$$\delta_i^s LB^s \leq s_i \leq \delta_i^s UB_i^s \quad \delta_i^b LB_i^b \leq b_i \leq \delta_i^b UB_i^b$$

  - Cardinality constraint to restrict the number of trades

$$\sum_{i=1}^{N} \delta_i^s + \delta_i^b \leq k$$

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Computational Study (8)

- Factor model (C factors) used to describe asset returns r_i :

$$r_i = \alpha_i + \sum_{c=1}^{C} \beta_{ic} f_c + \varepsilon_i$$

f_c	level of factor c
β_{ic}	sensitivity of asset i to factor c
α_i	mean return of asset i
ε_i	specific return of asset i

- The variance of returns is given by:

$$Var(r_i) = \sigma_i^2 = \sum_{c=1}^{C} \beta_{ic}^2 \sigma_{f_c}^2 + \sigma_{\varepsilon_i}$$

$\sigma_{f_k}^2$	factor variances
$\sigma_{\varepsilon_i}^2$	specific variances

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Computational Study (9)

- Tracking the target portfolio in terms of replicating the risk profile (vector of factor sensitivities)
- Sensitivity of the index to the factors given by I_c
- REBALANCE:

$$\begin{aligned}
&Min \quad \sum_{c=1}^{C} y_{P,c}^2 \sigma_c^2 + \sum_{i=1}^{N} x_i^2 \sigma_{\varepsilon_i}^2 \\
&s.t. \\
&\quad y_{P,c} = \left(\sum_{i=1}^{N} x_i \beta_{ic} \right) - I_c \qquad \forall c \\
&\quad \sum_{i=1}^{N} x_i \mu_i \geq \rho \\
&\quad \sum_{i=1}^{N} x_i = 1
\end{aligned}$$

$$\begin{aligned}
&x_i = n_i + b_i - s_i \\
&\delta_i^b LB_i^b \leq b_i \leq \delta_i^b UB_i^b \\
&\delta_i^s LB_i^s \leq s_i \leq \delta_i^s UB_i^s \\
&\delta_i^b + \delta_i^s \leq 1 \\
&\sum_{i=1}^{N} \left(\delta_i^b + \delta_i^s \right) \leq k \\
&x_i, b_i, s_i \geq 0 \\
&\delta_i^b, \delta_i^s = 0/1 \qquad \forall i
\end{aligned}$$

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Computational Study (10)

- Roundlot restrictions can easily be incorporated
- Additional constraints are in the model (trans. costs, duration, cash infusion I_c ...)

Empirical example:

- Models are implemented for 2 datasets from the fixed-income market
  - rebalancing a 20 bond portfolio, at most 6 trades, tracking an index of 330 bonds
  - rebalancing a 49 bond portfolio, at most 10 trades, tracking an index of 391 bonds
- 3 factors were used to describe return dynamics (explaining approx. 95% of total variance)
- residual risk term is ignored (focus is on factor risk)
- 200 points on the frontier are plotted for each problem

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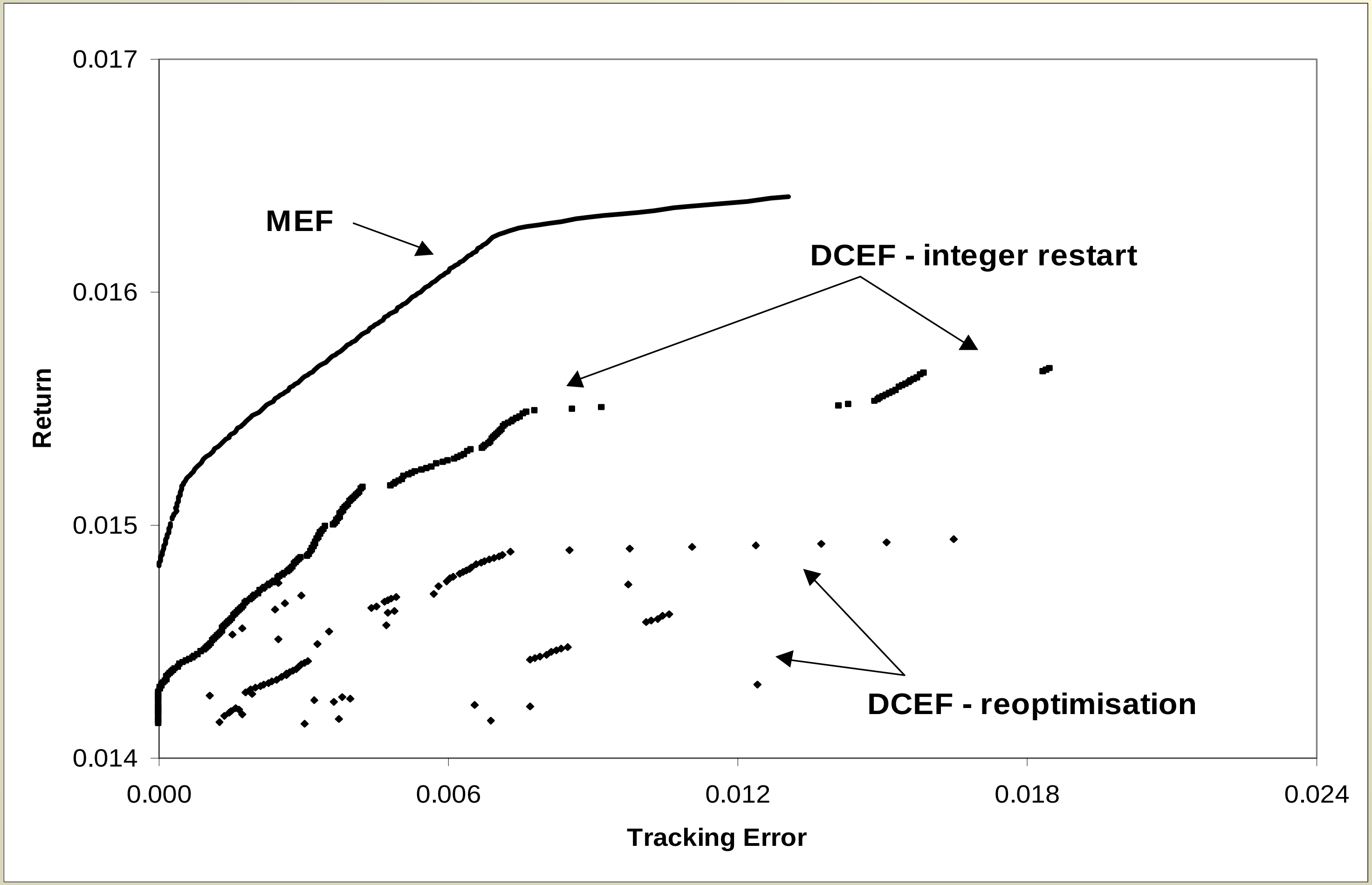
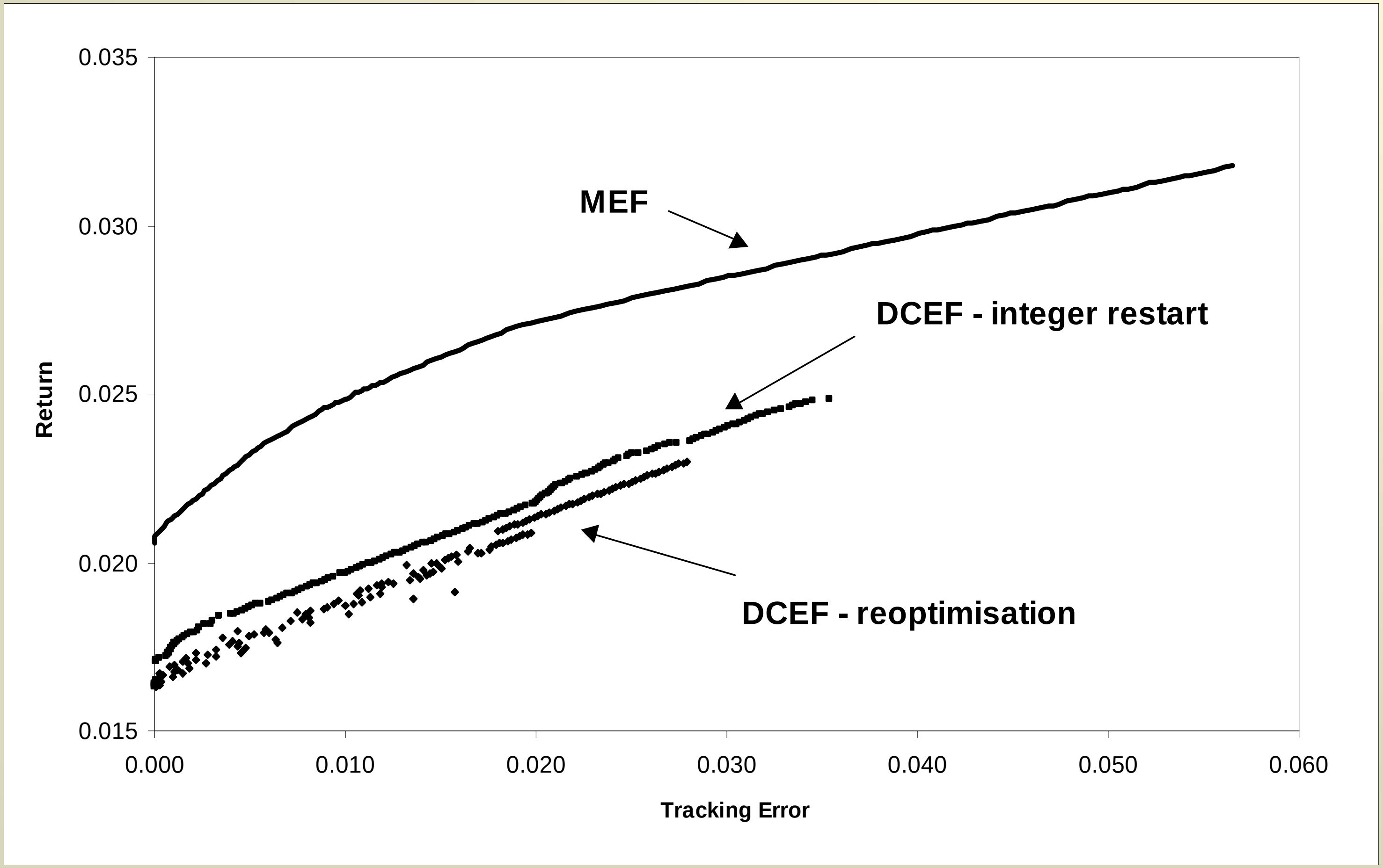
Computational Study (11)

- *Dataset 1 - 6/20 bond rebalancing*



Computational Study (12)

- *Dataset 2 - 10/50 bond rebalancing*



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Computational Study (13)

- Observations and Conclusions:
  - relative position of the MEF to the DCEF when constructing a new portfolio was determined by r_c
    - number of assets in the portfolio in the QP solution (MEF)
    - size of cardinality constraint
  - relative position of the MEF to the DCEF when rebalancing a given portfolio is determined
    - number of trades in the portfolio in the QP solution (MEF)
    - the initial holding
    - size of cardinality constraints
  - integer-restart heuristic seems to dominate the simple reoptimisation method

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Computational Study (14)

- Observations and Conclusions:
  - integer-restart method produces good sub-optimal L_c result, solver performance could be improved (e.g. pre-processing techniques)
  - effects of short sales and transaction costs (on the DCEF)
  - implementation of practical constraints in dynamic, multi-period asset management models
    - solving large-scale integer stochastic programs

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References:

- Chang, T.-J., N. Meade, J.E. Beasley and Y.M. Sharaiha (1999): *"Heuristics for cardinality constrained portfolio optimisation"*, to appear in Computers and OR

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