Handling Second-Order Stochastic Dominance through cutting-plane representations

Csaba I. Fábián
Eötvös L. University,
Kecskemét College,
visitor to CARISMA

Gautam Mitra
CARISMA,
Brunel University

Diana Roman
CARISMA,
Brunel University

Ackowledgement Dr Csaba Fábián's cooperation on the project was sponsored by OptiRisk Systems.

Brunel UNIVERSITY WEST LONDON

CARISMA

OptiRisk SYSTEMS

Introduction.

New portfolio-optimization models using the concept of Second-order Stochastic Dominance:

- SSD-constrained model, Dentcheva and Ruszczyński (2006),
- Multi-objective model, based on notion of *uniform dominance*, Roman, Darby-Dowman, and Mitra (2006).

These models lead to *polyhedral* convex programming problems,
i.e., objective / constraint functions are piecewise linear and convex.

Authors formulate linear programming problems by *lifting* the feasible sets into higher dimensions
i.e., by introducing new variables.

Solution approaches implemented by above authors:

- Special duality. Dual objects: utility functions.
  Regularized Decomposition adapted to dual problem.
- Direct approach.

Introduction continued.

Cutting-plane approach for special polyhedral convex problems

- Klein Haneveld and van der Vlerk (2006) for integrated chance constraints,
- Künzi-Bay and Mayer (2006) for conditional-value-at-risk objectives.

Former experience with cutting-plane approach:
proved *1-2 orders of magnitude* faster than direct approach using lifting representation (for large test problems).

We propose handling Second-order Stochastic Dominance through cutting-plane approach.

We present algorithmic description, implementation details, test results of cutting-plane approach for the multi-objective model of Roman, Darby-Dowman, Mitra.

Notation.

n : number of assets into which we may invest.

$r = (r_1, \ldots, r_n)^T$: returns of the different assets (random vector).

$x = (x_1, \ldots, x_n)^T$: portfolio; capital invested in different assets (decision variable).
Feasible portfolios: $x \in X$.

$w = r^T x$: yield of portfolio x.
(Random number in case decision is made on portfolio.)

$\hat{w}$: benchmark yield (random number). Possibly the yield of a benchmark portfolio.

Aim:

select a feasible portfolio whose yield distribution is preferable to benchmark yield distribution.

Brunel UNIVERSITY WEST LONDON CARISMA OptiRisk SYSTEMS

w preferable to $\hat{w}$: formulation with Second-order Stochastic Dominance.

Notation: $w \succeq_{SSD} \hat{w}$.

Economist's definition:

With any *suitable utility function*,

. *expected utility* of w is larger than or equal to *expected utility* of $\hat{w}$.

Formally: $E(U(w)) \geq E(U(\hat{w}))$ holds with any monotonic and concave utility function U.

Brunel UNIVERSITY WEST LONDON

CARISMA

OptiRisk SYSTEMS

w preferable to $\hat{w}$: formulation with Second-order Stochastic Dominance.

Assume discrete finite distributions with equally probable outcomes.

Let $w_1 \leq \ldots \leq w_S$ denote the ordered outcomes of w.

Let $\hat{w}_1 \leq \ldots \leq \hat{w}_S$ denote the ordered outcomes of $\hat{w}$.

Roman, Darby-Dowman, and Mitra :

$$w \succeq_{SSD} \hat{w} \quad \text{is equivalent to} \quad \underbrace{w_1 + \ldots + w_i}_{\text{tail}_i(w)} \geq \underbrace{\hat{w}_1 + \ldots + \hat{w}_i}_{\hat{\tau}_i} \quad \text{holding for} \quad i = 1, \ldots, S.$$

$r^T x \succeq_{ssd} \hat{w}$ is equivalent to $\text{tail}_i(r^T x) \geq \hat{\tau}_i \quad (i = 1, \ldots, S)$.

Multi-objective model Roman, Darby-Dowman, and Mitra :

$$\max_{x \in X} \left(\text{tail}_1(r^T x), \ldots, \text{tail}_S(r^T x) \right),$$

where the multi-objective maximization is considered with respect to the reference point $\hat{\tau} = (\hat{\tau}_1, \ldots, \hat{\tau}_S)$.

Motivation: *uniformly better* tails.

Putting into practice: Define and maximise achievement function. Simplest form:

$$\max_{x \in X} \Gamma(x), \qquad \text{where} \quad \Gamma(x) := \min_{1 \leq i \leq S} \left(\text{tail}_i(r^T x) - \hat{\tau}_i \right).$$

Brunel UNIVERSITY WEST LONDON   CARISMA   OptiRisk SYSTEMS

Computation of tails, using Rockafellar and Uryasev (2000) :

$$\text{tail}_i\left(r^T x\right) = \max_{t \in \mathbb{R}} \; it - \sum_{j=1}^{S} \left[t - r^{(j)T} x \right]_+ .$$

Lifting representation as linear programming problem:

$$\text{tail}_i\left(r^T x\right) = \quad \max \; it - \sum_{j=1}^{S} d_j$$

$$\text{such that} \quad t, \; d_1, \ldots, d_S \in \mathbb{R},$$

$$d_j \geq t - r^{(j)T} x, \quad d_j \geq 0 \qquad (j = 1, \ldots, S).$$

Using lifting representation, multi-objective model can be represented as LP having S^2 new variables.

In case of $10,000$ *scenarios, it means* 100 *million new variables.*

Computation of tails, using Rockafellar and Uryasev (2000) :

$$\text{tail}_i\left(r^T x\right) = \max_{t \in \mathbb{R}} \; it - \sum_{j=1}^{S} \left[t - r^{(j)T} x \right]_+ .$$

Cutting-plane representation using Künzi-Bay and Mayer (2006) :

$$\text{tail}_i\left(r^T x\right) = \min \sum_{j \in \mathcal{J}} r^{(j)T} x$$

$$\text{such that} \quad \mathcal{J} \subset \{1, \ldots, S\}, \quad |\mathcal{J}| = i.$$

Astronomical number of cuts!

But in a cutting-plane method, we only need a few of them. (Progressive cut generation)

Brunel UNIVERSITY WEST LONDON

CARISMA

OptiRisk SYSTEMS

Cutting-plane representation of the multi-objective problem:

$$\begin{array}{lll} \max & \vartheta & \\ \text{such that} & \vartheta \in \mathbb{R}, \quad x \in X, & \\ & \vartheta + \hat{\tau}_i \le \sum_{j \in \mathcal{J}_i} r^{(j)T} x & \text{for each} \quad \mathcal{J}_i \subset \{1, \dots, S\}, \ |\mathcal{J}_i| = i, \\ & & \text{where} \quad i = 1, \dots, S. \end{array}$$

Specialized cutting-plane method:

0. Initialize.

Set the stopping tolerance $\epsilon > 0$.

The initial cutting-plane model problem contains the constraints $x \in X$, and a single cut that we select arbitrarily. (Cut added to make objective bounded.)

1. *Solve model problem.*

Let $(\vartheta^*, \boldsymbol{x}^*)$ be an optimal solution of the current cutting-plane model problem.

Let $w^*_{j_1} \leq \ldots \leq w^*_{j_S}$ denote the ordered outcomes of $w^* = \boldsymbol{r}^T \boldsymbol{x}^*$.

Let $\mathcal{J}_i := \{j_1, \ldots, j_i\} \quad (i = 1, \ldots, S)$.

2. *Check for optimality.*

If

$$\vartheta^* + \hat{\tau}_i \leq \sum_{j \in \mathcal{J}_i} \boldsymbol{r}^{(j)T} \boldsymbol{x}^* + \epsilon \quad \text{holds for each} \quad i = 1, \ldots, S,$$

then $\boldsymbol{x}^*$ is an ϵ-optimal solution; stop.

If some of the above inequalities are not satisfied, then consider the violations

$$\vartheta^* + \hat{\tau}_i - \sum_{j \in \mathcal{J}_i} \boldsymbol{r}^{(j)T} \boldsymbol{x}^* \quad (i = 1, \ldots, S).$$

Let $\hat{\imath}$ $(1 \leq \hat{\imath} \leq S)$ denote the index that maximizes violation.

3. *Append cuts.*

Append the following cut to the cutting-plane model problem:

$$\vartheta + \hat{\tau}_{\hat{\imath}} \leq \sum_{j \in \mathcal{J}_{\hat{\imath}}} \boldsymbol{r}^{(j)T} \boldsymbol{x}.$$

Repeat from *step 1*.

Brunel UNIVERSITY WEST LONDON CARISMA OptiRisk SYSTEMS

Former computational study of multiobjective model   Roman, Mitra and Darby-Dowman

76 stocks from FTSE 100 index,

weekly/monthly observations during the period January 1993 - December 2003.

Scenario sets:

132 historical monthly returns, considered as equally probable scenarios.

565 historical weekly returns, considered as equally probable scenarios.

Benchmark distribution:

Type I : distribution of the FTSE 100 index.

Benchmark distribution is not SSD-efficient.

Optimal objective value is positive.

Optimal portfolio return distribution dominates benchmark distribution.

Type II : distribution of the stock with the highest expected return.

Benchmark distribution is SSD-efficient.

Optimal objective value is zero.

Optimal portfolio return distribution is the benchmark distribution.

Type III : 'ideal return distribution', composed of the individual optimal tails.

Benchmark distribution is not attainable.

Optimal objective value is negative.

Optimal portfolio return distribution comes uniformly close to benchmark.

Former computational study of multiobjective model continued

Solution approach:

lifting representation,

resulting LP problem solved by CPLEX.

Problems with 565 scenarios could not be solved in realistic time.

Present computational study of multiobjective model

Test problems

Same stocks as used by Roman, Mitra and Darby-Dowman in former study.

Further scenarios generated using Geometric Brownian motion (Ross 2002), paramers set on the basis of historical weekly returns.

Scenario sets of cardinality 5000, 7000, 10000.

Solution approach:

cutting-plane representation,

resulting LP problem solved by specialized cutting-plane method.

Each problem was solved in less then 25 seconds.

Brunel UNIVERSITY WEST LONDON

CARISMA

OptiRisk SYSTEMS

Implementation of cutting-plane approach for the multi-objective model

We use relative stopping tolerance,
and only add cuts where relative violations are significant.

Use of relative tolerance is justified from a decision maker's point of view,
and we found this approach more effective.

Structure of software developed:

- Cut generation implemented in C.
- Cutting-plane model problems formulated by
  AMPL modelling system (Fourer, Gay and Kernighan 1989),
  using AMPL COM Component Library (Sadki 2005).
- Solver: FortMP (Ellison, Hajian, Levkovitz, Maros, Mitra 1999).

Present computational study – test results

Type I benchmark distribution (distribution of the FTSE 100 index).

Numbers of iterations required to achieve different numbers of accurate digits in optimum:

	4	5	6	7
5000	26	26	19	19
7000	22	23	23	18
10000	25	28	28	28

Column headers show the prescribed accuracy in digits.
Row headers show the numbers of the scenarios.

Present computational study – test results

Type II benchmark distribution (distribution of the stock with the highest expected return)

Numbers of iterations required to achieve different numbers of accurate digits in optimum:

	4	5	6	7
5000	9	6	6	6
7000	9	9	9	9
10000	6	6	6	6

Column headers show the prescribed accuracy in digits.
Row headers show the numbers of the scenarios.

Present computational study – test results

Type III benchmark distribution ('ideal return distribution').

Numbers of iterations required to achieve different numbers of accurate digits in optimum:

	4	5	6	7
5000	12	14	19	20
7000	12	14	16	18
10000	14	16	19	21

Column headers show the prescribed accuracy in digits.
Row headers show the numbers of the scenarios.

Future prospects:

Technical improvement: Regularization.

Variance taken into account.

Two-stage generalization.

Regularization

Bundle-type method can be applied for the solution of the master problem, instead of pure cutting-plane method.

We propose to apply Level Method (Lemaréchal, Nemirovskii, Nesterov 1995).

Favorable experience with variants of Level Method adapted to stoch.progr.:

– Fábián and Szöke (2007),

– Ellison, Fábián, Mitra; talk on Thursday HB2 (11:30).

Variance taken into account

Mean-risk models using two risk measures (Roman, Mitra, Darby-Dowman 2006b):

- Portfolio characterized with *(expectation, variance, CVaR)* of yield.
- Approximation of efficient frontier constructed.
- Out-of-sample analysis: superior performance of non-extremal efficient portfolios.

Proposed extension of multi-objective model:

Given portfolio x, consider $\Gamma(x)$ as *efficiency measure* of x

$$\Gamma(x) := \min_{1 \le i \le S} \left(\text{tail}_i \left(r^T x \right) - \hat{\tau}_i \right)$$ (formerly defined as *achievement function*).

Characterize portfolio with *(variance of yield, Γ efficiency measure)*.

Construct approximation of efficient frontier using cutting-plane approach.

Brunel UNIVERSITY WEST LONDON CARISMA OptiRisk SYSTEMS

Two-stage SSD-models

2 time periods,

we can rebalance our portfolio at the beginning of each period,

let us compare *benchmark yield* and *portfolio yield* at the end of the second period.

Proposed solution methods

SSD-constrained model

cutting-plane approach, dual decomposition scheme, Fábián and Veszprémi (2007) :

- Variance terms inlcuded in objective,
  following ideas of Roman, Mitra, Darby-Dowman (2006).
- Strictly convex objective function,
  hence primal solution can be obtained through dual approach.

Multi-objective model

under construction.