Linear Gaussian Affine Term Structure Models with Unobservable Factors: Calibration and Yield Forecasting

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Outline

- Linear Kalman filter
- Linear Gaussian interest rate models
- Numerical experiments
- Nonlinear filter for term structure matching
- Summary

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Linear Kalman filter: basic theory

- Continuous time SDEs are often used to model dynamics of interest rates. Discretization of these equations may lead to variables which do not have a physical value.
- Two approaches:
  - Use "proxies" for unobserved variables (`e.g.` overnight interest rate/ 3-month LIBOR for short rate).
  - Estimate them from observed variables.
- Kalman filter is the main tool for the latter approach.
- Used in both model calibration and forecasting.

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Linear Kalman filter: the set-up

- Consider a discrete time, linear state space system

$$x_{n+1} = Ax_n + B + \epsilon_{n+1} \quad \textit{Evolution Equation}$$
$$y_n = Cx_n + D + w_n \quad \textit{Measurement Equation}$$

- ϵ_n, w_n zero mean, Gaussian and uncorrelated.
- $A, B, C, D, \mathbb{E}(\epsilon_n \epsilon_n^T) = Q > 0, \mathbb{E}(w_n w_n^T) = R \geq 0$ are constants.
- Only y_n is measured; x_n is of interest (not directly observable).

Kalman filter: the aims

- To estimate the conditional mean $\hat{x}_{n+1|n}$ (and possibly, $\hat{x}_{n+1|n+1}$);
- To estimate the variance of error; $P_{n+1|n} = \mathbb{E}(x_{n+1} - \hat{x}_{n+1|n})^2$;
- To estimate the parameters.

Kalman filter equations

$$
\begin{aligned}
\textit{innovations} \qquad & v_n = y_n - \underbrace{(C\hat{x}_{n|n-1} + D)}_{\hat{y}_n} \\
\textit{variance of innovations} \qquad & \Sigma_n = CP_{n|n-1}C^T + R \\
\textit{Kalman gain} \qquad & K_n = AP_{n|n-1}C^T\Sigma_n^{-1} \\
\textit{conditional mean} \quad & \hat{x}_{n+1|n} = A\hat{x}_{n|n-1} + B + K_n v_n \\
\textit{conditional variance} \quad & P_{n+1|n} = AP_{n|n-1}A^T + Q \\
& \qquad - AP_{n|n-1}C^T\Sigma_n^{-1}CP_{n|n-1}A^T
\end{aligned}
$$

Kalman filter equations (cont'd)

- Equations are recursive. $\hat{x}_{n+1|n}$ is conditional mean of unobserved variable; $P_{n+1|n}$ is its conditional variance.
- We need x_0, P_0 to start with.
- Even if noise terms are not Gaussian, K_n as above yields smallest error variance amongst all linear filters.
- Under certain conditions on A, $K_n \to K$ as $n \to \infty$ where K is a constant matrix.

Kalman Filter equations: applications

- **Calibration:** Maximise, over parameter vector θ, joint (Gaussian) probability function of observations:

$$L(y_n, \theta) = \sum_{i=n}^{T} \log p(y_n | \mathcal{F}_{n-1}, \theta)$$
$$= -\sum_{n=0}^{T} \left(\log \det(\Sigma_n) + v_n \Sigma_n^{-1} v_n \right) + \text{a constant}$$

  where T is number of samples.

- **Forecasting:** Given parameters, find K_n at t_n and hence find conditional mean of x_{n+1} at "next" time-step; $\hat{x}_{n+1|n}$ using the above equations. Also yields a predicted variance $P_{n+1|n}$.

Basic term structure models

- With continuous compounding at constant rate r, the price of 1 Euro payable at time T is

$$\lim_{N\to\infty}\left(1+\frac{r}{N}\right)^{-NT}=e^{-rT}.$$

- If, r_t is driven by a stochastic process, the price at time t is $\mathbb{E}\left(e^{-\int_t^T r_s ds}\right)$ (under correct measure).
- Dynamics of the continuously compounded interest rate r_t can be inferred from observed bond/ securities prices or *spot rates*.

Basic term structure models (cont'd)

- *Spot rates* R_{t,τ_k} are interest rates at time t for time to maturity τ_k.
- *Yield curve* (or spot rate curve) gives (static) relationship between different R_{t,τ_k} and τ_k, at a given t.
- *Short rate* at time t, r_t, is the instantaneous spot rate. This is a theoretical construct.
- In a linear Gaussian model (and certain other models), R_{t,τ_k} is an affine function of r_t.

Linear Gaussian term structure model with unobservable factors

We consider a two factor, linear Gaussian short rate model (Babbs-Nowman model)

$$r_t = \mu - \sum_{i=1}^{2} X_{i,t}$$

$$dX_{i,t} = -\alpha_i X_{i,t} + \sum_{j=1}^{i} \sigma_{ij} dz_{j,t}. \tag{1}$$

Here, r_t is the (unobservable) short rate, α_i, σ_{ij}, μ *etc* are (unknown) constants. $z_{i,t}$ are uncorrelated standard Wiener processes.

- Theoretical spot rates $\left(= \frac{-1}{\tau_k} \log \mathbb{E}\left[\exp\left(-\int_t^{t+\tau_k} r_s\, ds\right)\right]\right)$:

$$r_t(\tau_k|\theta) = A_0(\tau_k|\theta) + \sum_{i=1}^{2} A_i(\tau_k|\theta) X_{i,t}$$

where $A_i(\cdot|\cdot)$ are known functions, θ is the vector of parameters.

- Actual spot rates are "observed in noise":

$$R_{t,\tau_k} = r_t(\tau_k, \theta) + \epsilon_{t,k} \tag{2}$$

with $\mathbb{E}(\epsilon_{t,\tau_k}) = 0$, $\mathbb{E}(\epsilon_{t,k}^2) = h_k^2$.

- Given observed R_{t,τ_k} for various maturities at each t and (discretised) short rate equation given earlier, we can use Kalman filtering to calibrate this model and forecast short rate/ spot rates.

Numerical experiments I: data

- Weekly data on 7 UK gilt yields (= government bond spot rates, published daily) from January 2001 to January 2006 from *Datastream*.
- 261 observations; 150 used for calibration and 100 for validation.
- 15 parameter, 2 factor linear model calibrated using Kalman filter +QML.

Numerical experiments II: parameter values

Parameter values			
α_1	0.7036	α_2	0.2807
σ_{11}	0.0080	σ_{22}	0.1806
λ_1	0.4249	λ_2	-0.0414
μ	0.2768	h_1	0.0006
h_2	0.0007	h_3	0.0014
h_4	0.0017	h_5	0.0011
h_6	0.0007	h_7	0.0010
σ_{12}	-0.0003		

$h_1^2, \cdots, h_7^2$ are measurement noise variances; λ_i are prices of risk.

Numerical experiments III: relative errors in yield prediction

- Measure of error: sample mean of the relative absolute error

$$\frac{|\text{observed yield-predicted yield}|}{\text{observed yield}}$$

  over the relevant set of observations (either in-sample or out-of sample).

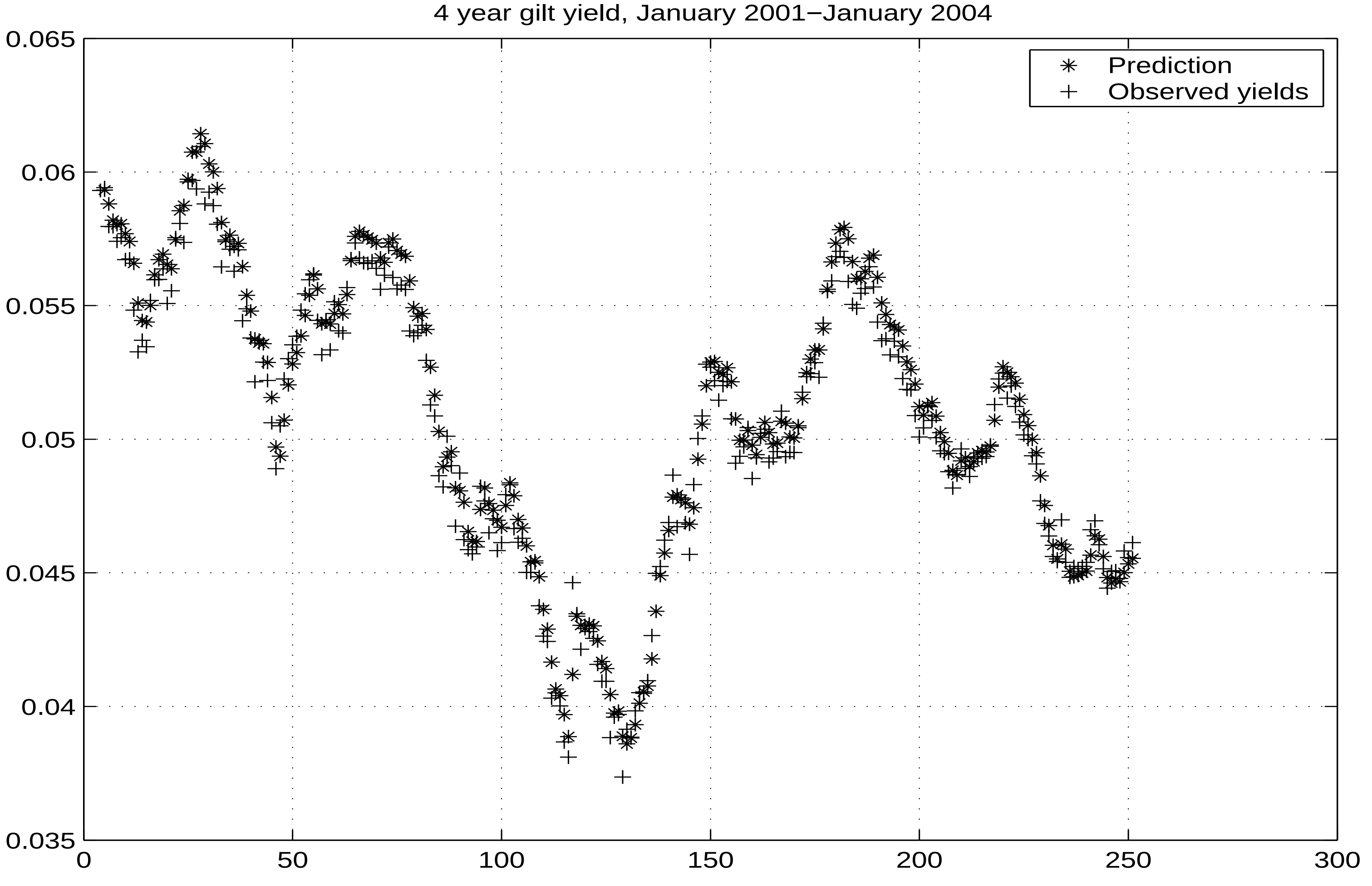
- Computed for one step ahead prediction of yields.

- Maximum out-of-sample error (over 7 yields) is only 2.31%, over a period of almost two years from calibration.

Mean relative absolute errors

yields	in-sample	out-of-sample
$3m$	0.0111	0.0118
$6m$	0.0178	0.0081
$1y$	0.0331	0.0216
$2y$	0.0337	0.0229
$4y$	0.0262	0.0152
$8y$	0.0209	0.0231
$10y$	0.0161	0.0213

Numerical Experiments IV: sample gilt yield plot



Samples from 150 onwards are post-calibration data.

Other numerical experiments

- Principal component analysis: 95% variation is explained by first two factors.
- One factor model with full or restricted yield spectrum: One factor is adequate if we use only short dated yields or only long dated yields.
- 4-step ahead prediction: Both one factor and two factor models perform similarly.

Nonlinear filter for term structure matching I: assumption

- Apart from yield prediction, another relevant problem is yield curve matching (*i.e.* finding values of $\hat{X}_{i,t_n|t_n}$ which best explain yield vector y_n at time t_n).
- For a calibrated model with parameter vector $\hat{\theta}$, we now assume that measurement noise is unknown-but-bounded:

$$|R_{t,\tau_k} - r_t(\tau_k|\hat{\theta})| < \delta_{t,k} R_{t,\tau_k}$$

  for some bounded $\delta_{t,k} > 0$.
- This does not change yield formulae; measurement noise variances do not affect bond pricing equations.

Nonlinear filter for term structure matching II: LP problem

- Now consider the problem at each time-step t

$$\min_{X_{i,t}} \max_{k} \frac{|R_{t,\tau_k} - r_t(\tau_k|\hat{\theta})|}{R_{t,\tau_k}}$$

- This is a linear programming problem in the state variables $X_{i,t}$; can be solved at each time-step to find $\hat{X}_{i,t|t}$ (instead of using a Kalman state update).

Nonlinear filter for term structure matching III: results

- Worst case mean relative absolute error improves from 2.67% (Kalman state update) to 2.22% (linear prog. update) in term structure matching; deteriorates from 2.31% (Kalman update) to 2.83% (lin. prog. update) in one step ahead prediction.
- We can also update μ at each step simultaneously with the state update ($r_t(\tau_k|\hat{\theta})$ is affine in μ).
- Consistent, *by definition*, to underlying model (unlike Nelsen-Siegel type curves or splines).

Summary

- Extensive numerical evidence presented on short term prediction ability of linear Gaussian term structure models with unobservable states.
- A two state model can perform very well out-of-sample in short term yield prediction.
- A new, simple nonlinear filter to improve term structure matching proposed; provides a term structure curve which is consistent with the assumed dynamics.
- Applications: pricing, scenario generation, bond portfolio optimization.