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Generating Asset Return Scenarios using Copula Method

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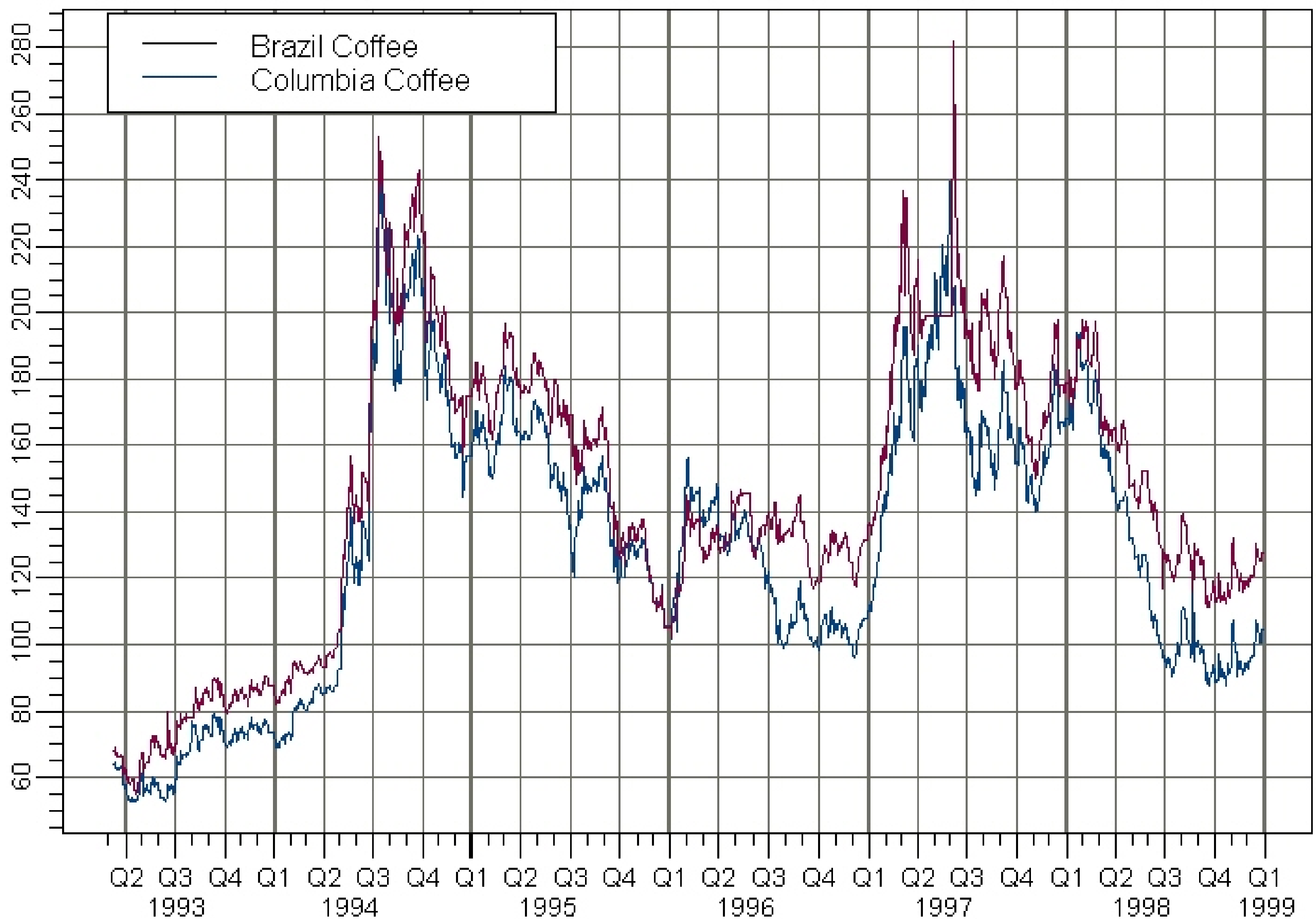
Motivation Example

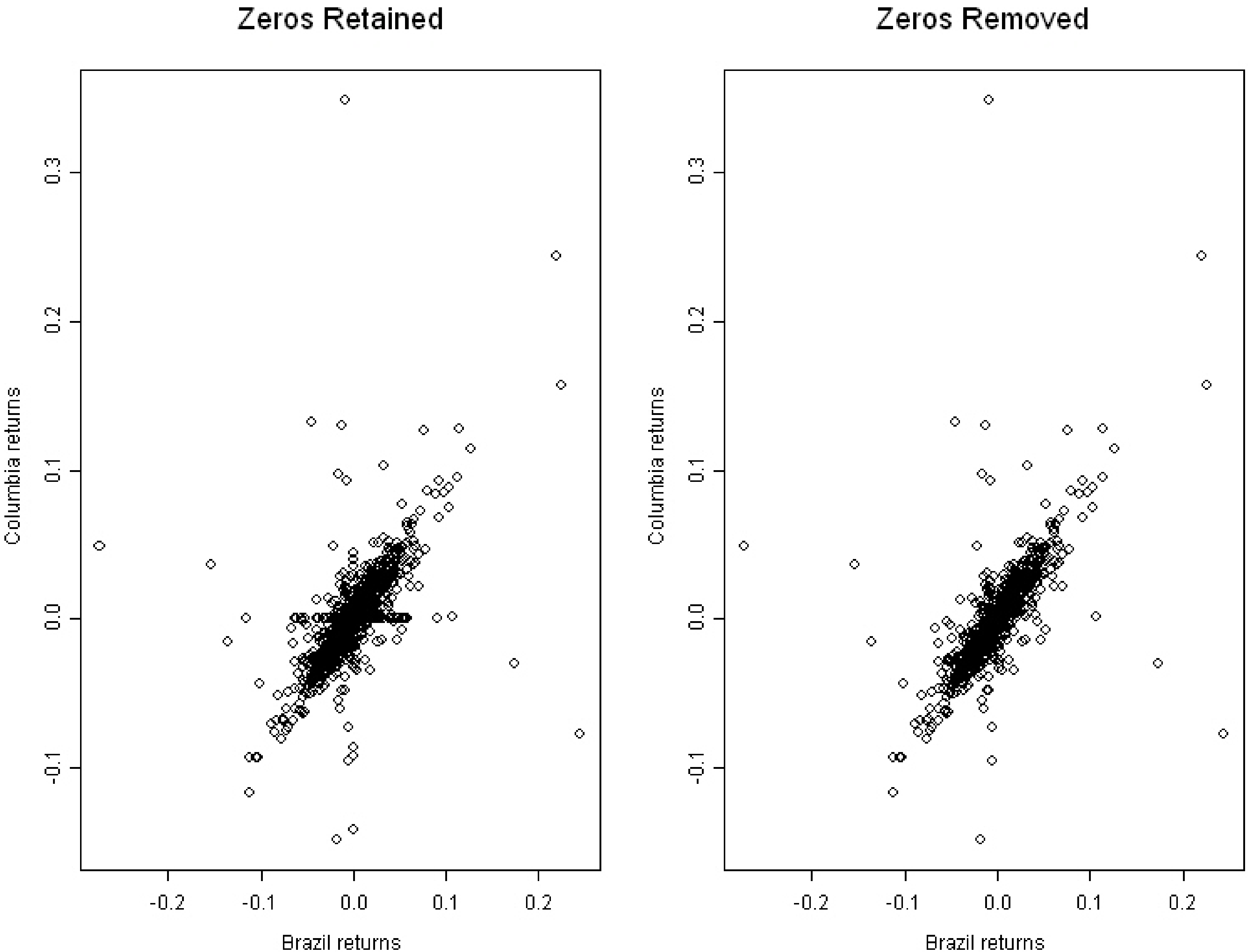
Some coffee in the morning!

“Mathematician is a machine that turns caffeine into theorems”

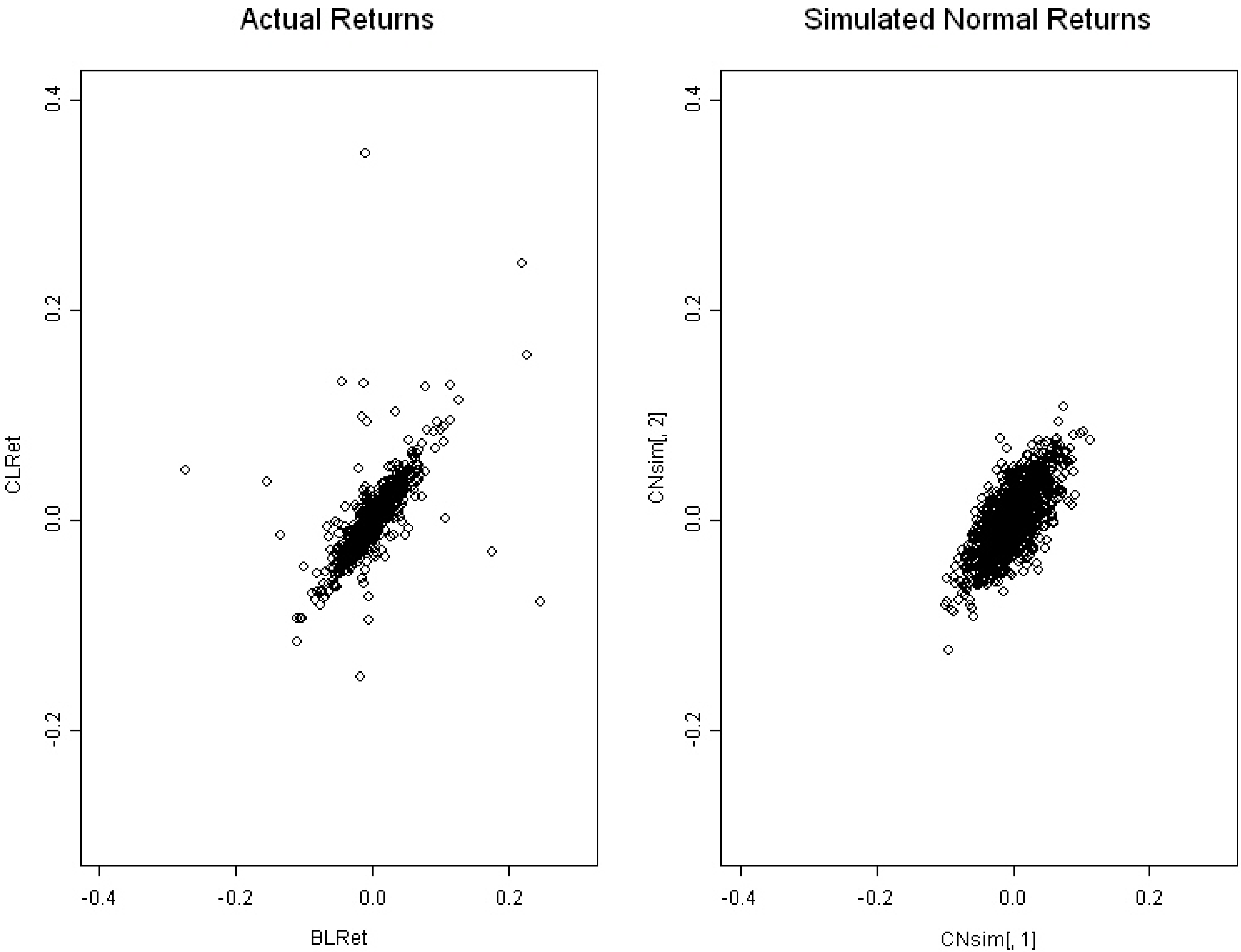
--Paul Erdos

Daily Spot Price of Coffees





Zeros Retained
Zeros Removed
Columbia returns
Brazil returns
-0.2
-0.1
0.0
0.1
0.2
0.3


Actual Returns
CLRet
BLRet
0.4
0.2
0.0
-0.2
-0.4
-0.2
0.0
0.2
Simulated Normal Returns
CNsim[, 2]
CNsim[, 1]
0.4
0.2
0.0
-0.2
-0.4
-0.2
0.0
0.2

Motivations:

- We want to generate a set of returns for future scenarios.

  Assuming we dismiss normality and generalise our simulation methodology in two main aspects:

  - We aim to allow marginal distributions that could take any arbitrary form.
  - We want to relax the modelling of dependence beyond the concept of correlation.

Problems

- How can we glue arbitrary return distributions together and still maintain their dependence structure?
- How can we model tail dependence and still keep the same marginal distributions?

- The answer to these questions is copulas!

- What is the Copula?

---The way in which random variables relate to each other.

Starter

- Brief Historical Background
  - 1940's: Hoeffding studied properties of multivariate distributions
  - 1959: The word "Copula" appeared for the fist time (Sklar 1959)
  - 1998: Academic literature on how to use copulas in risk management (Embrechts et al. 1999)
  - 2004: Some financial/insurance companies have started to use copulas as a risk management tool

$$X, Y \quad r.v.;$$

$$F(x), F(y)$$

$$F(x) = P[X \leq x], F(y) = P[Y \leq y]$$

$$H(x, y) = P[X \leq x, Y \leq y]$$

	b_1	b_2	b_3	…	b_k
a_1	$p_{1,1}$	$p_{1,2}$	$p_{1,3}$	…	$p_{1,k}$
a_2	$p_{2,1}$	$p_{2,2}$	$p_{2,3}$	…	$p_{2,k}$
…	…				…
…	…				…
a_m	$p_{m,1}$	$p_{m,2}$	$p_{m,3}$	…	$p_{m,k}$

$P(A=a_i \text{ and } B=a_j):=p_{i,j}$

Copula Function

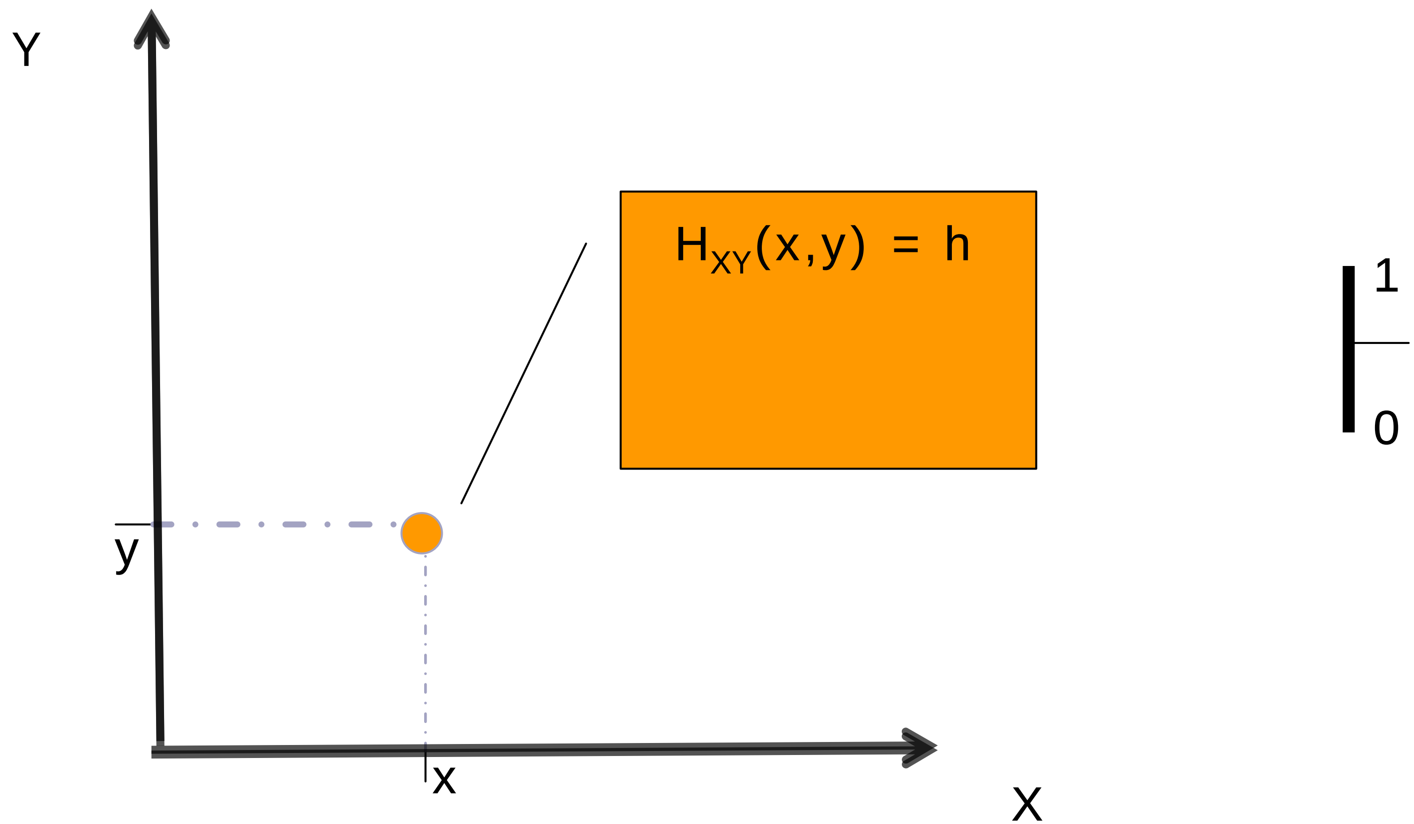
A copula is a multivariate distribution whose marginals are all uniform over (0,1)

$$C(u_1,\ldots,u_d) = F(F_1^{-1}(u_1), F_2^{-1}(u_2),\ldots, F_d^{-1}(u_d))$$

Sklar Theorem

$$H(x_1,\ldots,x_d) = C(F_1(x_1), F_2(x_2),\ldots, F_d(x_d))$$

Copula Function



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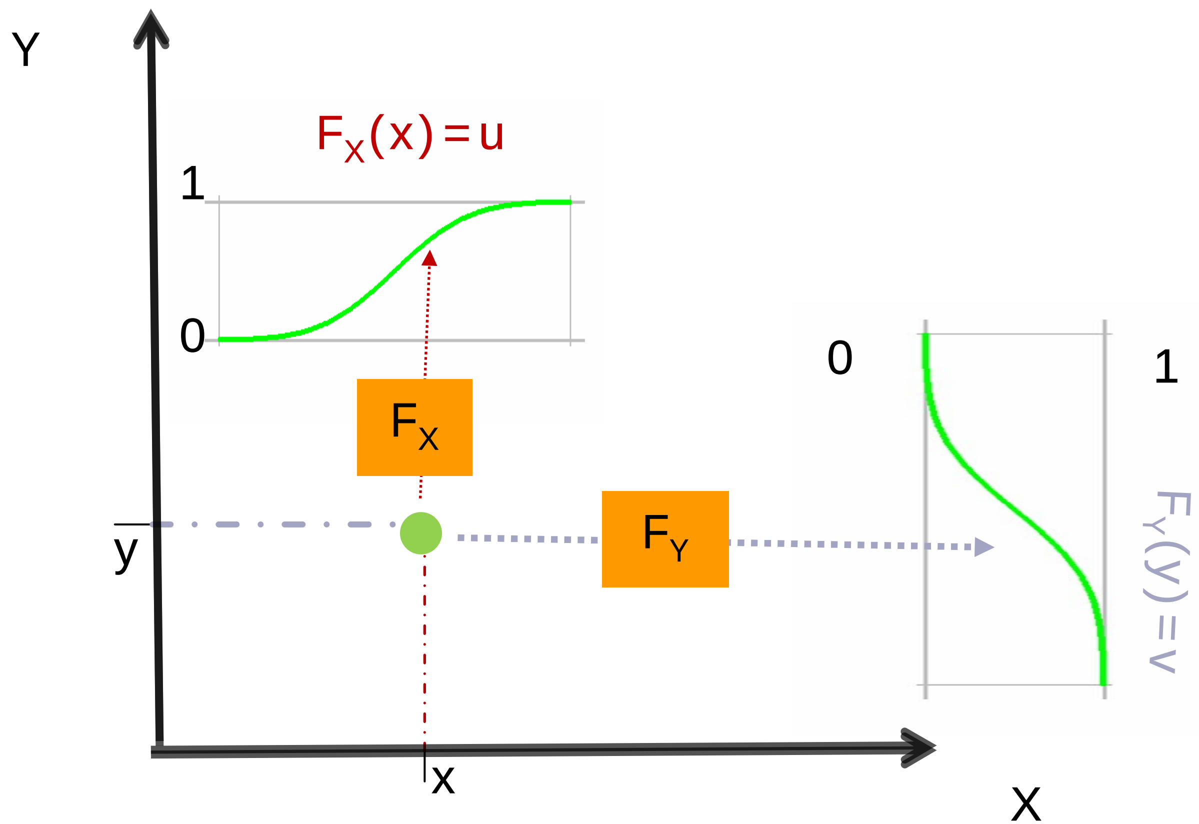
Copula Function



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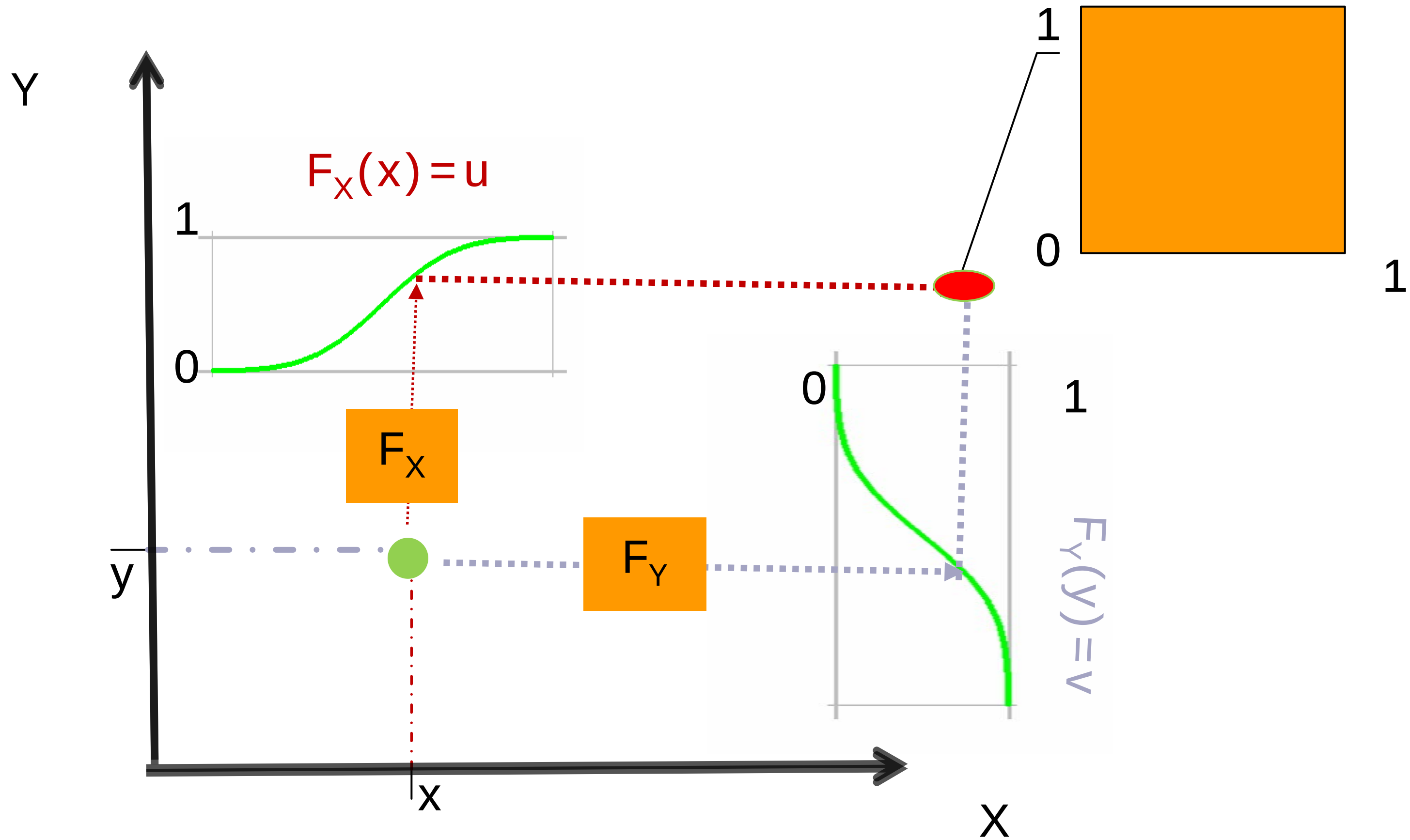
Copula Function



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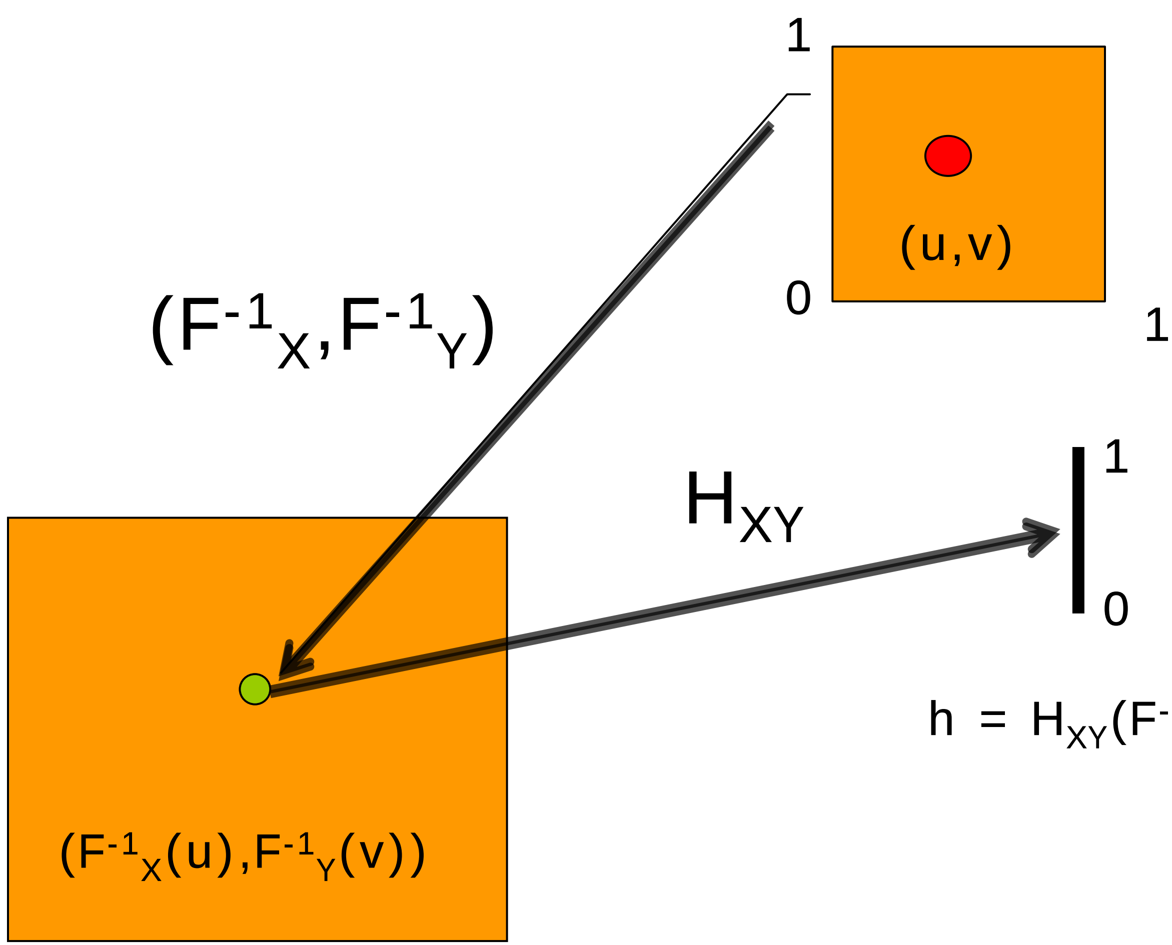
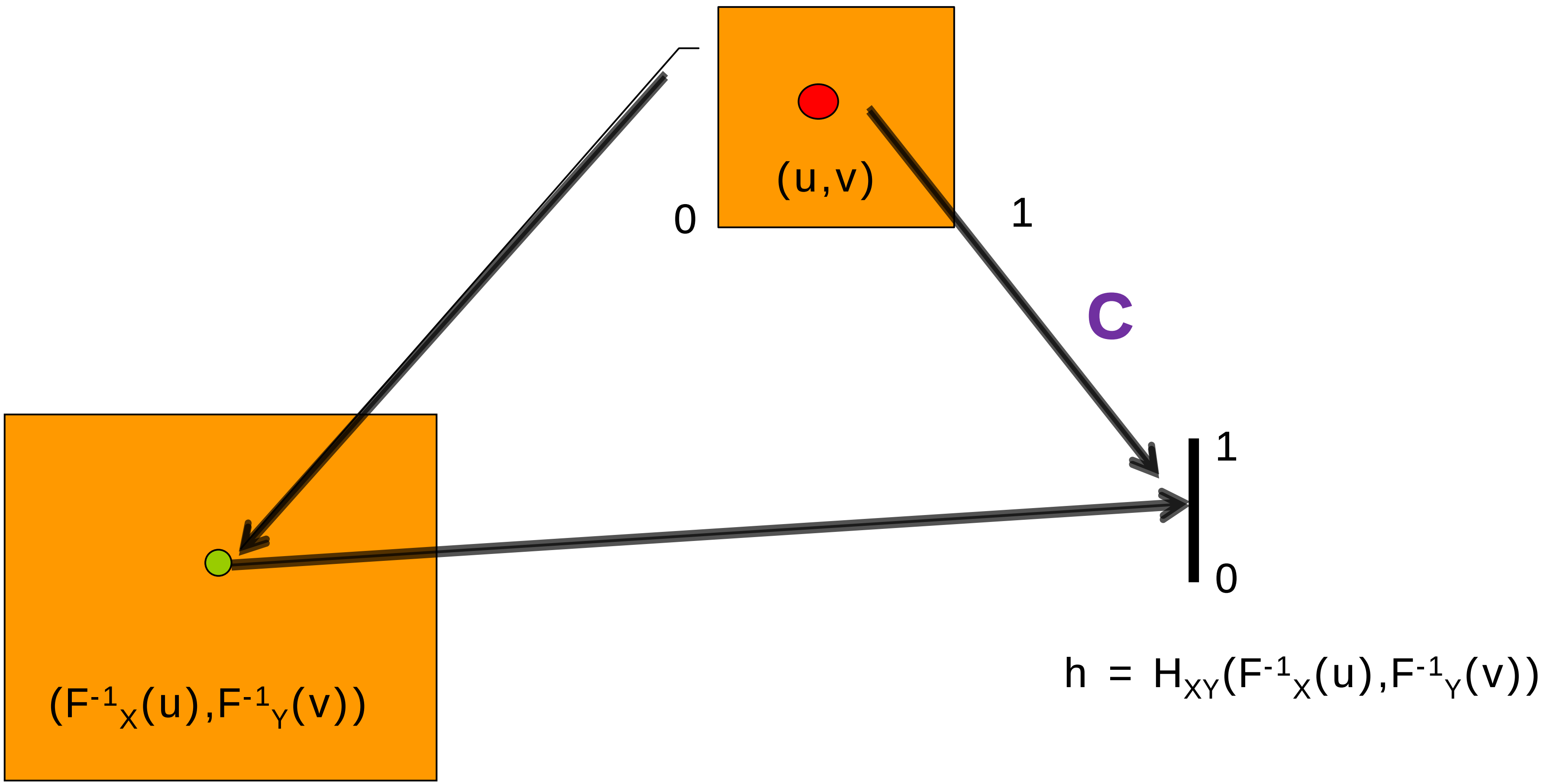
Copula Function



$h = H_{XY}(F^{-1}_X(u), F^{-1}_Y(v))$

Copula Function



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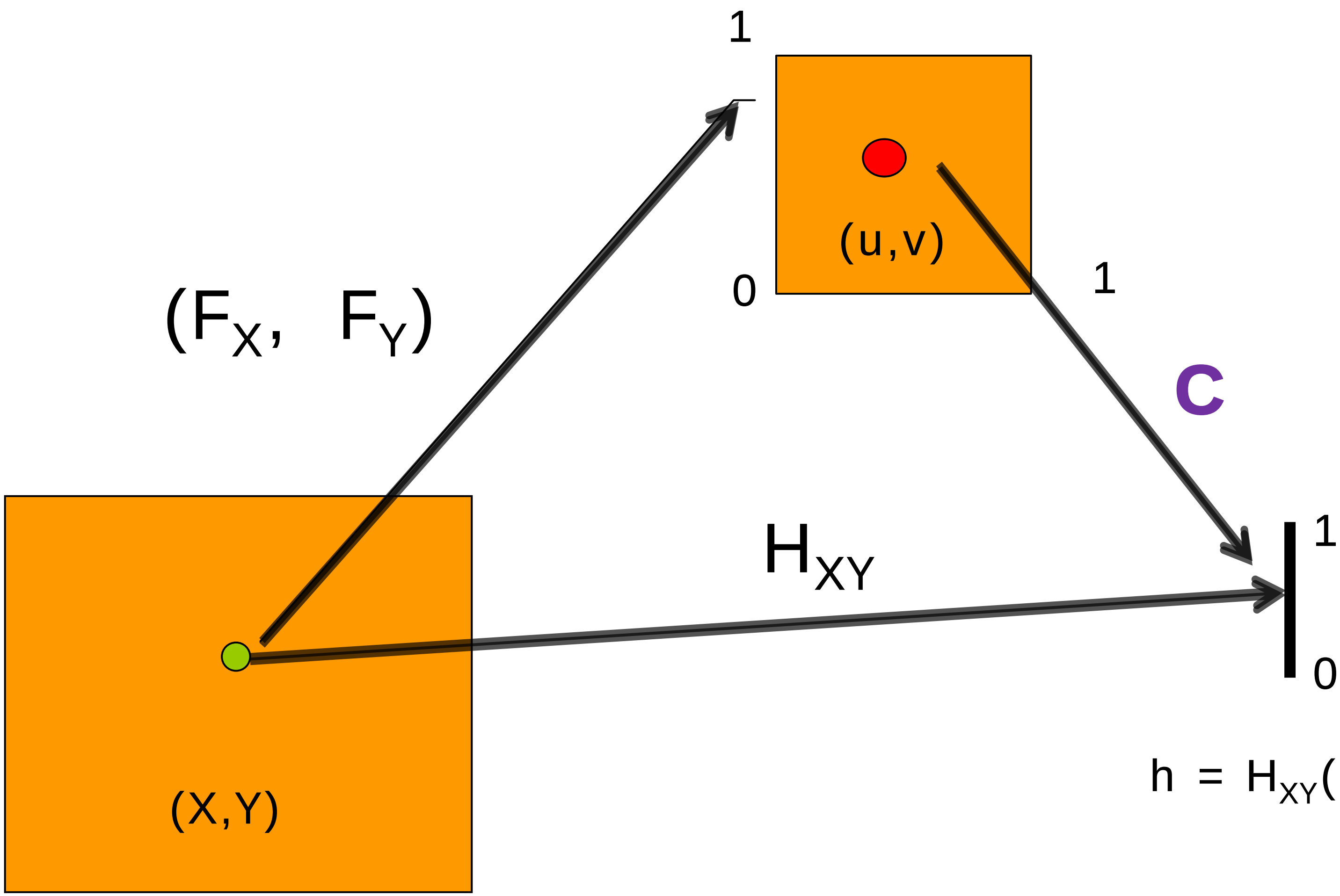
$$C(u,v) = F(F^{-1}(u), F^{-1}(v))$$



$h = H_{XY}(F^{-1}_X(u), F^{-1}_Y(v))$

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Example of Copulas

- Gaussian Copula
  with gaussian bivariate distribution

$$N\left[\begin{pmatrix}0\\0\end{pmatrix},\begin{pmatrix}1 & \rho\\ \rho & 1\end{pmatrix}\right]$$

$$C_{\rho}(u,v)=\Phi_{\rho}[\Phi^{-1}(u),\Phi^{-1}(v)]$$

where

Φ is the cdf of the univariate stadard normal distribution

Φ_{ρ} is the cdf of the bivariate standard normal with correlation coefficient equal to ρ

$$C_{\rho}(u,v)=\int_{-\infty}^{\Phi^{-1}(u)}\int_{-\infty}^{\Phi^{-1}(v)}\frac{1}{2\pi\sqrt{1-\rho^{2}}}\exp\left\{\frac{-(s^{2}-2\rho st+t^{2})}{2(1-\rho^{2})}\right\}dsdt$$

T Copula

- Suppose that X~$t_d(v, 0, \rho)$ where v is the degree of freedom and ρ is a correlation matrix. The t copula is the unique copula of X

$$C^t_{v,\rho}(u) = \phi_\rho(t_v(X_1) \le u_1, \ldots, t_v(X_d) \le u_d) = t_{v,\rho}(t_v^{-1}(u_1), \ldots, t_v^{-1}(u_d))$$

- where t_v is the df of a standard univariate t distribution and $t_{v,\rho}$ is the joint distribution function of the vector X~$t_d(v, 0, \rho)$ and P is a correlation matrix; in two dimensions we use ρ to represent the correlation coefficient.

- Monte Carlo Simulation

$$u_s \sim uniform(0,1)$$

$$r_{i,s} = F_{r_i}^{-1}(u_s)$$

- Uniform random numbers

$$(u_1,\ldots,u_n)$$

- We can then calculate the marginals from

$$F_{r_1}^{-1}(u_1),\ldots,F_{r_n}^{-1}(u_n)$$

$$F(r_1,\ldots,r_n)$$

$$F(r_1,\ldots,r_n) = C(F_{r_1},\ldots,F_{r_n})$$

- By using copula method we can separate the univariate margins and the multivariate dependence structure. This proves to be very convenient in scenario generation.

- Estimate the copula function and the marginals…

Copula Estimation

- Nonparametric Copula Estimation
  This means estimating both copula and marginals nonparametrically.

- Semi-parametric Copula Estimation
  This means estimating copula parametrically and marginals nonparametrically.

- Parametric Copula Estimation
  This means estimating copula and marginals with parametrical distribution families.

Exchange rates levels

- > udeu
- V1 V2 V3 V4
- 1 1/13/1993 Wed 0.82994 -0.0058507450
- 2 1/20/1993 Wed 0.81913 -0.0131106080
- 3 1/27/1993 Wed 0.81120 -0.0097281690
- 4 2/3/1993 Wed 0.84451 0.0402419450
- 5 2/10/1993 Wed 0.85034 0.0068796920
- .
- .
- .
- 670 11/16/2005 Wed 0.85486 0.0055954870
- 671 11/23/2005 Wed 0.85074 -0.0048311540
- 672 11/30/2005 Wed 0.85058 -0.0001880890
- 673 12/7/2005 Wed 0.84822 -0.0027784340
- 674 12/14/2005 Wed 0.83446 -0.0163552270
- 675 12/21/2005 Wed 0.84181 0.0087695270
- 676 12/28/2005 Wed 0.84359 0.0021122590

- The generalized extreme value distribution has cumulative distribution function

$$F(x;\mu,\sigma,\xi) = \exp\left\{-\left[1+\xi\left(\frac{x-\mu}{\sigma}\right)\right]^{-1/\xi}\right\}$$

```
> gev.fit(udeu[,4])
$conv
[1] 0

$nllh
[1] -2096.158

$mle
[1] -0.004137744  0.010695222 -
0.219991515
```

- > udata<-pgev(udeu[,4],-0.004137744, 0.010695222, -0.219991515)
- > udata
- [1] 0.310220171 0.115373162 0.193955938 0.999984767 0.732757263 0.142051992
- [7] 0.578993885 0.762846581 0.806337804 0.386874173 0.120180369 0.107314578
- [13] 0.380147579 0.170437690 0.510042400 0.100425218 0.496974735 0.950392736
- [19] 0.703820563 0.449646472 0.315223805 0.859522662 0.878107694 0.992632619
- [25] 0.500101462 0.741843191 0.840120239 0.314183211 0.938774817 0.542186545
- [31] 0.606336751 0.073287592 0.352343272 0.214349782 0.004302800 0.220004760
- …………………

Test

- We will use the copula approach to picture four mixtures of normals together, maintaining a specified correlation structure and modelling (symmetric) tail dependence according to a t-copula.
- To simulate a t-copula with v degrees of freedom, we have to proceed the following steps:

Algorithms

- Find the Cholesky decomposition $C_{cholesky}$ of the correlation matrix C (dimension n * n)
- Draw a vector of n standard normals *u* and calculate $C_{cholesky}$ *u*.
- Draw from $s \sim \chi_v^2$ and multiply the result of the second step by $\sqrt{v} / \sqrt{s}$ i.e., calculate

$$x = C_{Cholesky} u \frac{\sqrt{v}}{\sqrt{s}}$$

- Each element of *x (X1,...,Xn)* is inserted into the cumulative distribution function to arrive at uniformly distributed variables $u_i \sim t_v(x_i)$
- Repeat steps 2 – 4 (m) times.

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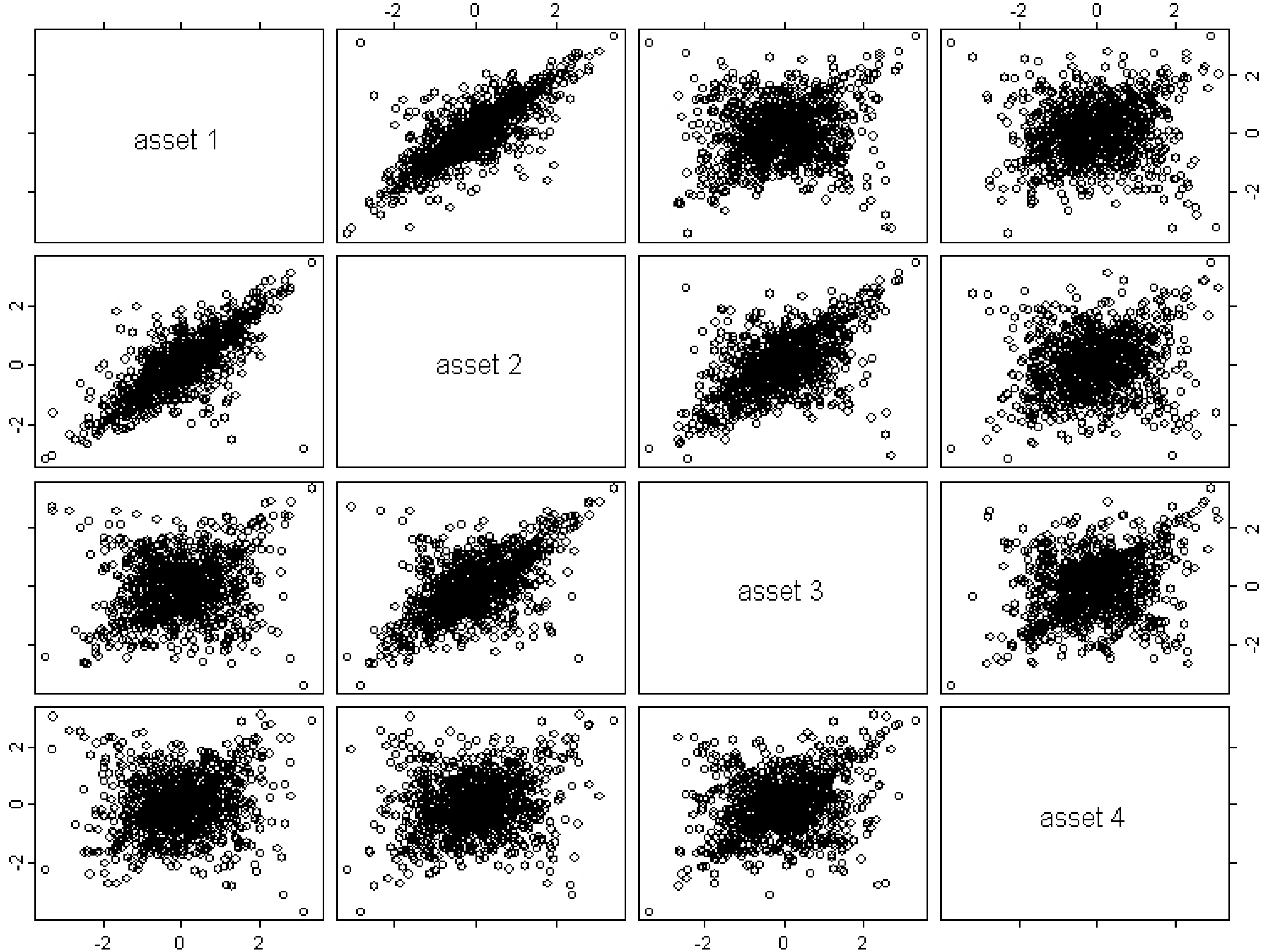

- Graph 1: T Copula with 2 Degrees of Freedom and Standard Normal Margins

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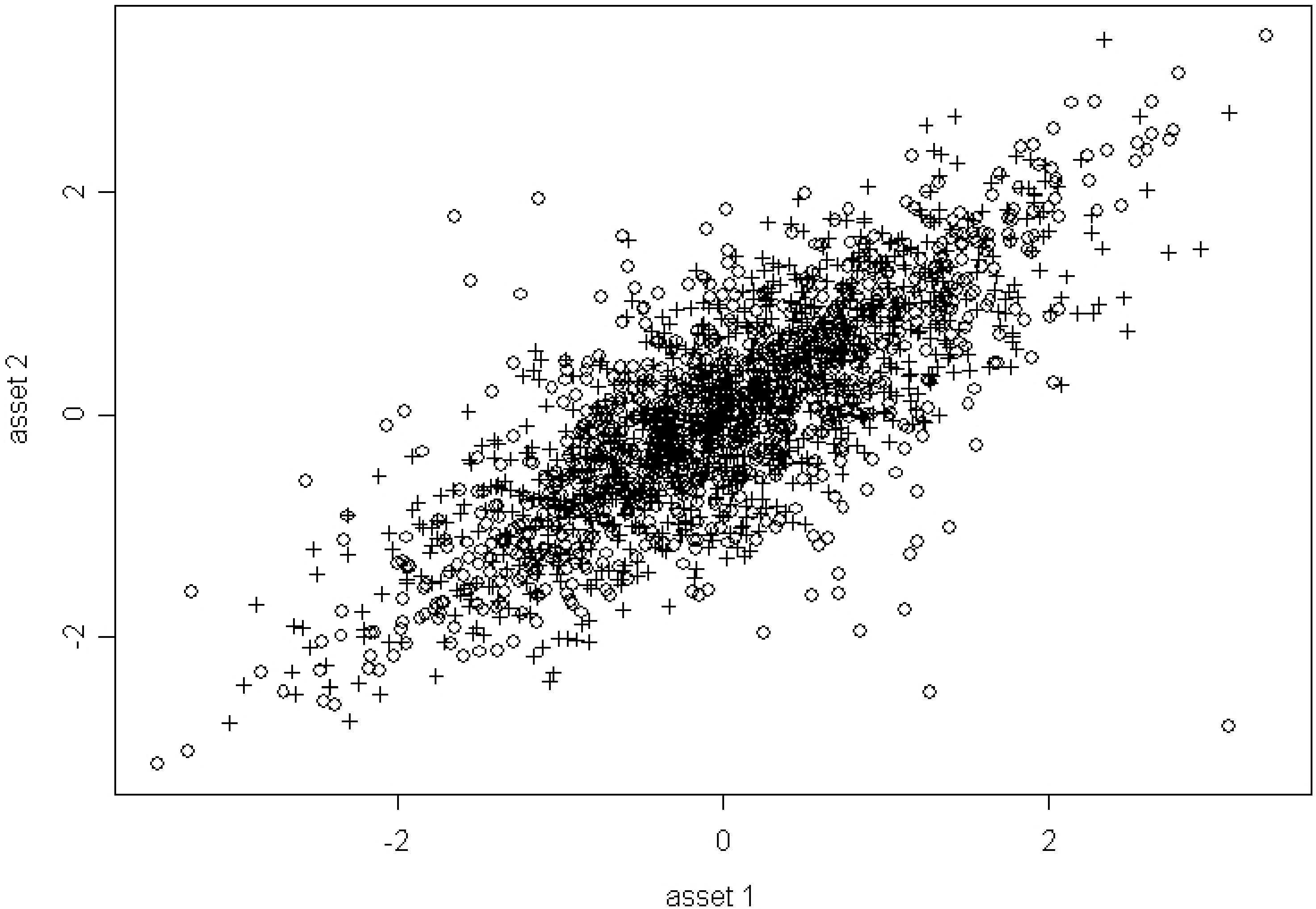
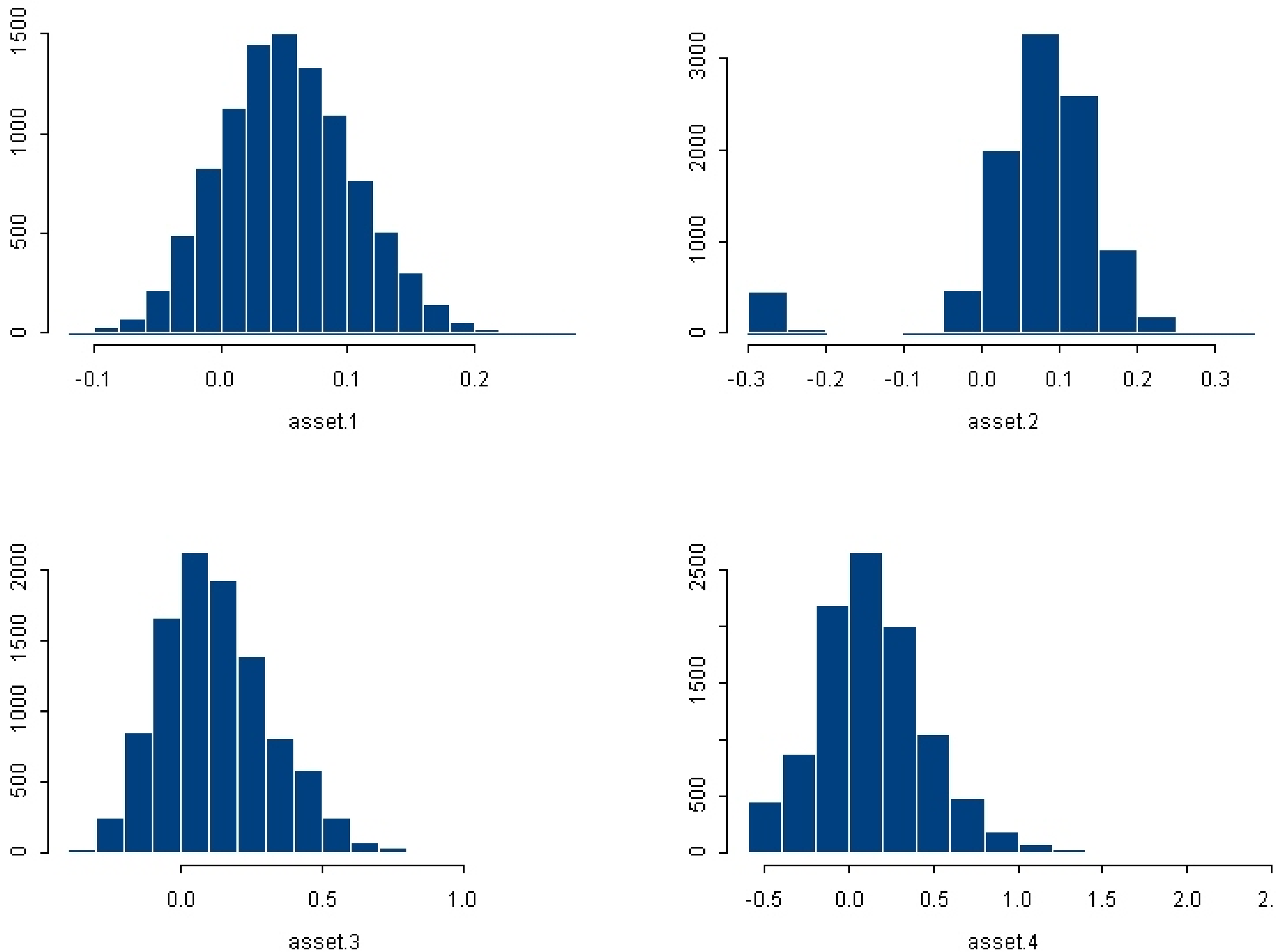


Graph 3

- Scatter plot for Normal Distribution versus t copula
- + t-copula; o standard normal



- **Graph 4: Marginal Distributions in Four-Asset Test Case.**

- Marginal distributions are glued together with the use of a t-copula and stored in the scenario matrix S.
- Now we can code a scenario-based Markowitz optimisation, where portfolio variance is calculated using all scenarios for each weight allocation rather than by supplying a single covariance matrix.
- We can then find the solutions for return points along the efficient frontier.

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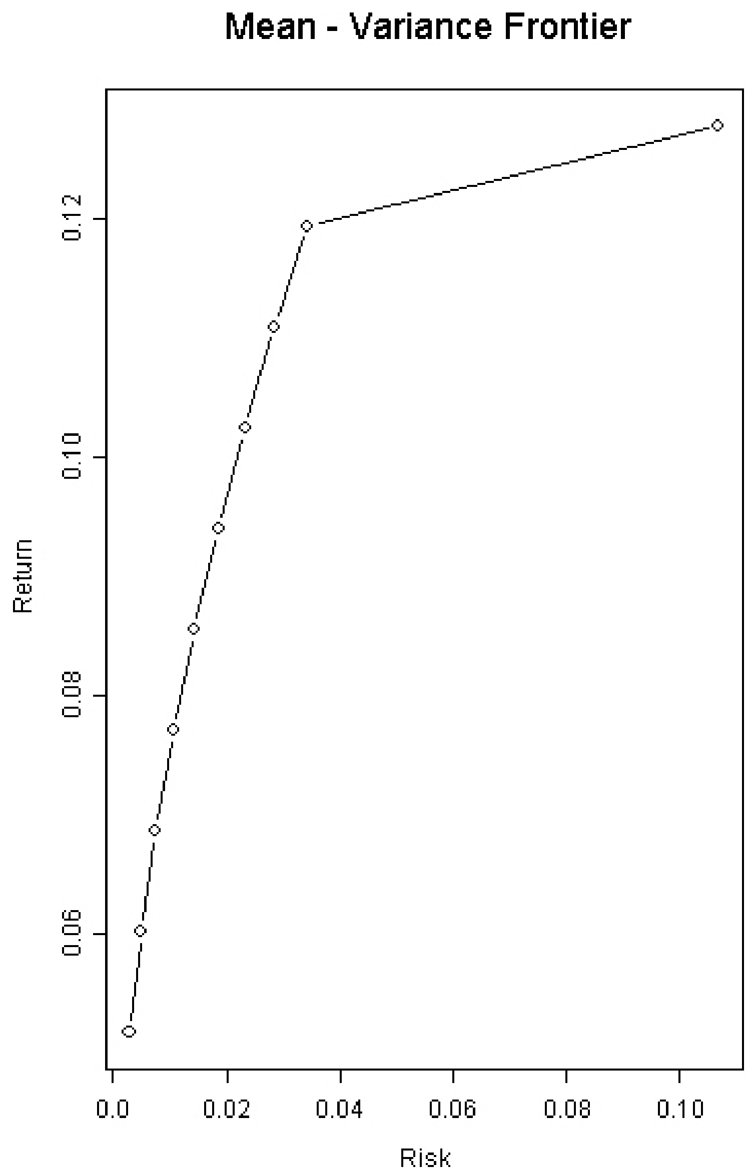

- It shows the solutions for ten return points along the efficient frontier. We can use these solutions as a reference point for the following scenario optimisation.

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- Next stage:
  - Implement CVaR model
  - Using ICA/CCA

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Thank You !

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