PORTFOLIO CONSTRUCTION BASED ON STOCHASTIC DOMINANCE AND TARGET RETURN DISTRIBUTIONS

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STRUCTURE

1. **Preliminaries**

   a) The portfolio selection problem

   b) Review of models for choice under risk

2. **Motivation and contribution**

3. **A model for finding a SSD efficient portfolio close to a specified distribution**

   a) Second order stochastic dominance (SSD) and multi-criteria optimisation

   b) The reference point method

4. **Computational results**

5. **Concluding remarks**

The portfolio selection problem: a particular case of the general problem of deciding between 2 random variables when larger outcomes are preferred

- An initial amount of capital to invest, n assets, each asset j giving a return R_j at the end of the investment period (R_j random variable)

- If x_j is the proportion of wealth invested in asset j, the portfolio return is $R_X=x_1R_1+\dots+x_nR_n$ (R_X random variable).

- For another portfolio resulting from choice $(y_1,\dots,y_n)$, the portfolio return is the random variable $R_Y=y_1R_1+\dots+y_nR_n$

- The problem: how do we choose between R_X and R_Y?

The portfolio selection problem

- **Scenario generation, historical data → discrete random variables**

- T scenarios; p_i=probability of scenario i occurring; $p_1+p_2+\ldots+p_T=1$

- Data: r_{ij}=return of asset j in scenario i; so R_j finitely distributed over $\{r_{1j}, r_{2j},\ldots,r_{Tj}\}$

- The portfolio return R_X finitely distributed over $\{R_{X1},\ldots,R_{XT}\}$ where $R_{Xi}=x_1r_{i1}+\ldots+x_nr_{in}$ is the return in scenario i

Models for choice under risk

-Mean-risk models

-Expected utility maximisation

-Stochastic dominance

Aims:

-Define a preference relation among random variables

-Find random variables that are nondominated with respect to that preference relation

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Models for choice under risk: Mean-risk models

•2 scalars attached to a r.v.: the ***mean*** and the value of a ***risk measure***.

•A risk measure ρ : a function mapping random variables into real numbers.

•In the mean-risk approach with risk measure given by ρ, ***R_X is preferred to r.v. R_Y*** (or ***R_X dominates R_Y***) if $E(R_X) \geq E(R_Y)$ and $\rho(R_X) \leq \rho(R_Y)$ with at least one strict inequality.

•Non-dominated r.v. (portfolios): via an optimisation problem

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Models for choice under risk: Expected utility maximisation (EUM)

•One scalar attached to a r.v.: **its expected utility.**

•**2 random variables are compared using their expected utility values: the larger value is preferred.**

•<u>A utility function</u>: a real valued function U defined on real numbers (representing possible wealth levels).

•<u>Expected Utility Theory:</u> extends a utility function defined on real numbers to a utility function defined on random variables.
An utility value E[U(X)] is assigned to each random variable X:

$$E[U(X)] = \int U(x)dF(x)$$

where F is the cumulative distribution function of X.

Expected utility maximisation (EUM)

- What conditions should a utility function for capital satisfy such that EUM leads to **rational decisions**?

risk attitude	form of utility function
non-satiation (rationality)	non-decreasing
+risk-aversion	+concave

- **Non-satiation**: non-arguable
- **Risk aversion**: accepted as the common attitude in investment

A utility function which is ***non-decreasing and concave*** corresponds to **observed economic behaviour**.

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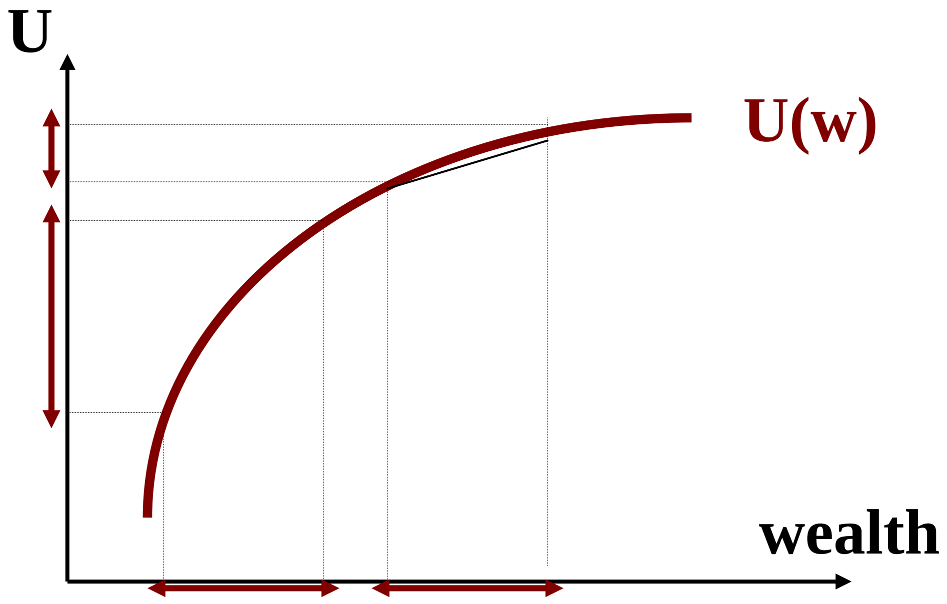
Expected utility maximisation (EUM)



Risk-aversion: a surplus of wealth is more valuable at lower wealth levels $\rightarrow$ concave utility function

Problem: how to choose a <u>specific</u> utility function?.

Models for choice under risk: Stochastic dominance (SD)

SD ranks choices (random variables) under assumptions about general characteristics of utility functions (eliminates the need to explicitly specify a utility function).

Random variables are compared by pointwise comparison of some performance functions constructed from their **cumulative distribution** functions.

Models for choice under risk: Stochastic dominance (SD)

Connection with EUM:

-1st order stochastic dominance (FSD):

R.v. X is preferred to r.v. Y with respect to FSD ($X >_{FSD} Y$) $\Leftrightarrow$:

$E[U(X)] \geq E[U(Y)]$, $\forall U$ non-decreasing utility function.

This means that X is preferred to Y by all rational investors.

-2nd order stochastic dominance (SSD):

R.v. X is preferred to r.v. Y with respect to SSD ($X >_{SSD} Y$) $\Leftrightarrow$

$E[U(X)] \geq E[U(Y)]$, $\forall$ U non-decreasing and concave.

This means that X is preferred to Y by all rational investors who are risk averse.

Pros and cons of models for choice under risk

1. **Mean-risk models:**
- Convenient from a computational point of view
- May lack a theoretical, rational basis for selection.The validity of the results provided is questionable.

2. **Expected utility models:**
- How to choose a utility function?

3. **Stochastic dominance models:**
- The most sound (theoretical) basis for making choice under risk
- Very difficult from a computational point of view

2. Motivation and contribution

The attempts to combine the practicality of mean-risk models with the theoretical solid basis of stochastic dominance $\rightarrow$ mean-risk models consistent with SD.

Open questions:

- Which risk measure to choose?
- Which solution to implement? Only criterion of choice: the level of trade-off between the mean return and the risk.

Motivation and contribution

The proposed model that finds a portfolio which is:

- non-dominated with respect to SSD ($\Leftrightarrow$ optimal for every rational and risk averse investor).
- has a return distribution close to a **reference** (target, goal, aspiration) distribution which is introduced as input data.

The whole distribution of portfolio return is taken into account, not just summary statistics.

This is achieved using a computationally tractable procedure which uses linear inequalities and LP [Goal] models.

Motivation and contribution

Close to a reference distribution means "***close or better***":

3 cases:

1. The reference distribution is SSD efficient → portfolio whose return distribution is the reference distribution.

2. The reference distribution is not SSD efficient → portfolio whose return distribution is SSD efficient - better than reference distribution (<u>outperforms the goal!</u>)

3. The reference distribution is not attainable (too high outcomes) → portfolio whose return distribution is SSD efficient and comes close to the reference distribution.

Motivation and contribution

How the model is created:

1. The SSD efficient solutions are (Pareto) non-dominated solutions of a multi-objective linear model (Ogryczak 2002).

2. To obtain a specific solution which yields a distribution of returns close to the reference distribution, use reference point method (Wierzbicki 1983)

So: the reference point method is embedded into the multi - objective linear model in order to produce meaningful solutions for the portfolio selection in an interactive way.

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SSD efficiency and multiple criteria optimisation

The portfolio selection problem: <u>case of T equally probable scenarios.</u>

portfolio x → return ($f_1(x),\ldots,f_T(x)$):
random variable seen as a <u>T-dimensional vector</u>, where

$f_i(x) = \sum_{j=1}^{n} r_{ij} x_j$ (the portfolio return in scenario i=1,…,T)

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SSD efficiency and multiple criteria optimisation

How do we express that $z^1=(f_1(x^1),\ldots f_T(x^1))$ dominates $z^2=(f_1(x^2),\ldots f_T(x^2))$ with respect to SSD relation?

The sum of the worst k outcomes of z^1 is greater than the the sum of the worst k outcomes of z^2, $\forall k=1,\ldots,T$.

Equivalently: the CVaR of z^1 at confidence level k/T is smaller than the the CVaR of z^2, at confidence level k/T, $\forall k=1,\ldots,T$.

Example:

(1,4,3,2) and (3,5,0,2) $\xrightarrow{\text{Order the outcomes}}$ (1,2,3,4) and (0,2,3,5)

$\xrightarrow{\text{Cumulate the outcomes}}$ (1,3,6,10) and (0,2,5,10) $\xrightarrow{\text{Pareto comparison}}$

(1,4,3,2) dominates (3,5,0,2) with respect to SSD

SSD efficiency and multiple criteria optimisation

The sum of the worst k outcomes of a vector $z=(z_1,\dots,z_T)$ is the optimal value of a LP problem:

$$\text{Max} \quad kt_k - \sum_{i=1}^{m} d_{ki}$$

$$\text{Such that:} \quad t_k - z_i \le d_{ki} \qquad \text{for } i=1,\dots T$$

$$d_{ki} \ge 0 \qquad \text{for } i=1,\dots T$$

(Uryasev and Rockafellar 2000, Ogryczak 2002)

t_k is the k-th worst outcome of the vector z

SSD efficiency and multiple criteria optimisation

The set of SSD non-dominated solutions is the set of Pareto non-dominated solutions of the multi-objective LP:

$$V \max\left(t_1 - \sum_{i=1}^{T} d_{1i}, 2t_2 - \sum_{i=1}^{T} d_{2i}, \ldots, Tt_T - \sum_{i=1}^{T} d_{Ti}\right) \qquad (1)$$

Such that:

$$t_k - f_i(x) \le d_{ki} \qquad \text{for i, k=1,…T}$$

$$d_{ki} \ge 0 \qquad \text{for i, k=1,…T}$$

$$\sum_{j=1}^{n} x_j = 1, x_j \ge 0 \qquad \text{for j=1,…n.}$$

where: $f_i(x) = \sum_{j=1}^{n} r_{ij} x_j$ is the portfolio return in scenario i.

The k-th objective function is the sum of the worst k outcomes of the portfolio return

The reference point method

The set of non-dominated solutions of (1) is infinite.

How do we choose a specific solution?

The Reference Point Method: generalised goal programming methods, used in Multi-objective optimisation for selecting a specific Pareto efficient solution

Goal programming: specify a target (goal) in the objective space and try to come close to it

Better than goal programming: come "**close or better**" – if the target is not efficient, outperforms it $\rightarrow$ quasi-satisficing decisions (Wierzbicki 1983)

The reference point method

Choose T desired values (aspiration levels, reference points), one for each objective function.

Multi-objective $\rightarrow$ single objective: A special scalarizing achievement function is constructed. When optimised (maximised), it generates a Pareto non-dominated solution of our model, so a SSD efficient solution for the portfolio problem.

The reference point method

Let $z^*=(z_1^*,\ldots,z_T^*)$ be the target (aspiration levels)

Reference points for cumulative outcomes → reference points for ordered outcomes → reference distribution.

The simplest form of scalarizing achievement function:

$$\gamma_{z^*}(z) = \min_{1\le i\le T}(z_i - z^*_i) + \varepsilon\sum_{i=1}^{T}(z_i - z^*_i) \qquad (2)$$

where ε>0 is a small positive parameter.

The terms $(z_i-z^*_i)$ in the definition of (2) are usually replaced with more complicated functions $\gamma_i(z_i,z^*_i)$: partial achievement functions

The portfolio optimisation model

Maximise $\delta + \varepsilon(\sum_{k=1}^{T} z_k - \sum_{k=1}^{T} ref_k)$

Subject to:

$z_k = kt_k - \sum_{i=1}^{T} d_{ki}$ for k=1,…,T

$z_k - ref_k \geq \delta$ for k=1,…,T

$t_k - \sum_{j=1}^{n} x_j r_{ij} \leq d_{ki}$ for k=1,…,T; i=1,…,T

$\sum_{j=1}^{n} x_j = 1$

$x_j \geq 0$ for j=1,…,n

$d_{ki} \geq 0$ for k=1,…,T; i=1,…,T

- $ref_1,\ldots,ref_T$ are given: the aspiration levels for $z_1,\ldots,z_T$
- Variables $t_1,\ldots,t_T$: the ordered outcomes of portfolio return
- Variables $z_1,\ldots,z_T$: the cumulative outcomes; $z_i=t_1+t_2+\ldots+t_i$

The portfolio optimisation model

Interpretation of results:

1. optimal value of the objective function>0 ⇔ the reference distribution is not SSD efficient →we obtain a portfolio whose return distribution is <u>better</u> (SSD efficient)

2. optimal value of the objective function=0 ⇔ the reference distribution is SSD efficient →we obtain a portfolio whose return distribution is <u>exactly</u> the reference distribution

3. optimal value of the objective function<0 ⇔ the reference distribution is unattainable → we obtain a portfolio whose return distribution comes <u>close</u> to the reference distribution

The portfolio optimisation model

Refinement: One could introduce also reservation levels: $z^r=(z_1^r,\ldots,z_T^r)$ (i.e. should be achieved if at all possible: first level target) → partial achievement functions $\gamma_i(z_i, z^*_i, z_i^r)$.

For example, piece-wise linear partial achievement functions:

$$\gamma_{z*}(z) = \frac{\alpha(z_i - z^r{}_i)}{z*_i - z^r{}_i} \quad \text{for } z_i \leq z^r{}_i$$

$$\gamma_{z*}(z) = \frac{z_i - z^r{}_i}{z*_i - z^r{}_i} \quad \text{for } z^r{}_i \leq z_i \leq z^*{}_i$$

$$\gamma_{z*}(z) = \frac{\beta(z_i - z*_i)}{z*_i - z^r{}_i} + 1 \quad \text{for } z_i \geq z^*{}_i$$

with α, β such that $0<\beta<1<\alpha$.

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4. Computational results

Data set:

- 20 assets from Hang Seng index ($n=20$)
- 131 time periods, considered as equally probable scenarios ($T=131$)
- Monthly returns: r_{ij}, $i=1,\ldots,T$; $j=1,\ldots,n$

We consider $\varepsilon=0.00005$.

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Computational results

Case 1: improve on a distribution

The reference distribution: Hang Seng index for the same time periods.

Result: positive optimum (0.185) $\rightarrow$ the reference distribution is not SSD efficient and the model improves on it.

α=0.18 means that the cumulated outcomes of the portfolio produced with our model (P1) are greater than the cumulated outcomes of the index by at least 0.18.

Computational results

Case 1: improve on a distribution. In-sample analysis

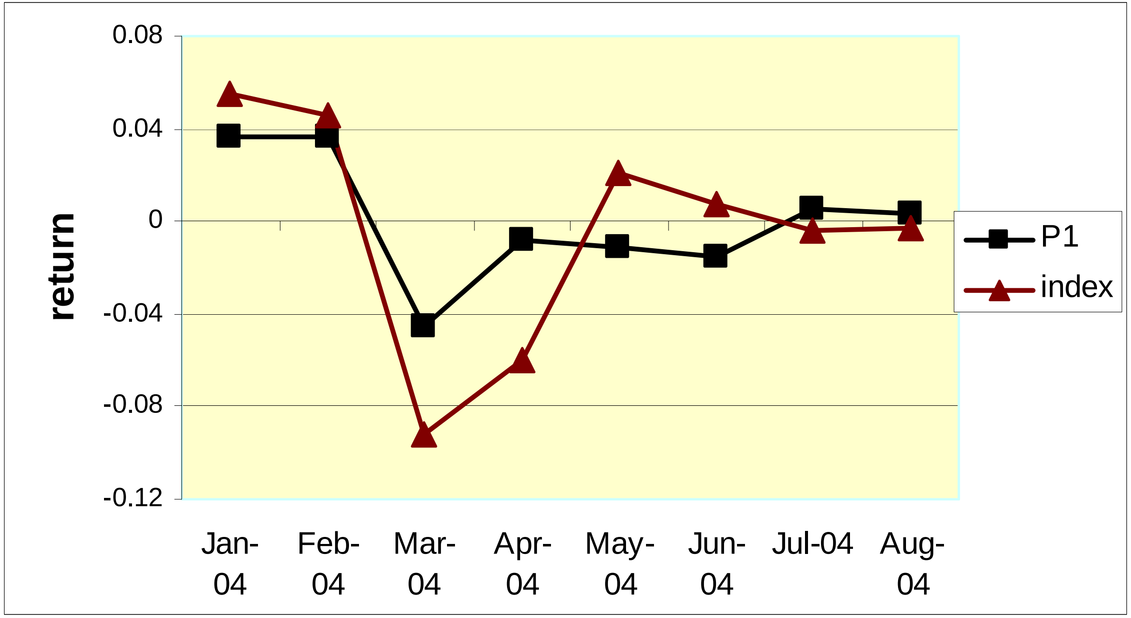
	P1	index
Mean	0.0073	0.0059
Median	0.0117	0.0094
St. Deviation	0.0635	0.0861
Skewness	0.0240	-0.0864
Kurtosis	0.9901	2.1539
Range	0.3890	0.6128
Minimum	-0.1688	-0.3482
Maximum	0.2202	0.2645

In-sample parameters for the return distribution of the index and the portfolio obtained with our model (P1)

P1 has a return distribution with better statistics than the index (higher mean, higher median, lower variance, higher skewness, lower range, better minimum, etc.)

Computational results

Case 1: improve on a distribution. Out-of-sample analysis



The performance of the index and P1 for the next 8 time periods following the date of selection (January 2004-August 2004).

Our portfolio P1 has less variation as compared to the index, thus avoiding extreme low values.

Computational results

Case 2: a SSD efficient reference distribution

The reference distribution: the mean-variance efficient portfolio with the highest expected return (precisely, the distribution of the asset with the highest expected return)

This distribution is SSD efficient (and efficient in any mean-risk model).

Result: a zero optimal value and the solution portfolio is consisting of just one asset: the one with highest mean return).

This example serves only for demonstrative purposes.

Computational results

Case 3: an unattainable reference distribution

The outcomes of the reference distribution: the optimal values for the 131 cumulated outcomes in the portfolio selection problem.

Result: a negative optimum (-0.55)→ the aspiration levels form an unattainable distribution

(These was to be expected: no feasible portfolio could have a return distribution with such high outcomes.)

The solution portfolio of our model (P3) has a return distribution that comes close to the reference distribution (decrease some cumulative outcomes until feasibility)

Computational results

Cases 1-3 : Summary of in-sample results

	P1	index	P3	Max_mean
Mean	0.0073	0.0059	0.0086	0.0127
Median	0.0117	0.0094	0.0083	0.0160
St. Deviation	0.0635	0.0861	0.0679	0.1433
Skewness	0.0240	-0.0864	-0.1563	-1.4026
Kurtosis	0.9901	2.1539	1.0257	5.7481
Range	0.3890	0.6128	0.4291	0.9981
Minimum	-0.1688	-0.3482	-0.2109	-0.7263
Maximum	0.2202	0.2645	0.2182	0.2718

In-sample parameters for the return distribution of the index and portfolios P1, P3 and Max-mean

(Max_mean: the portfolio consisting of the asset with the highest expected return)

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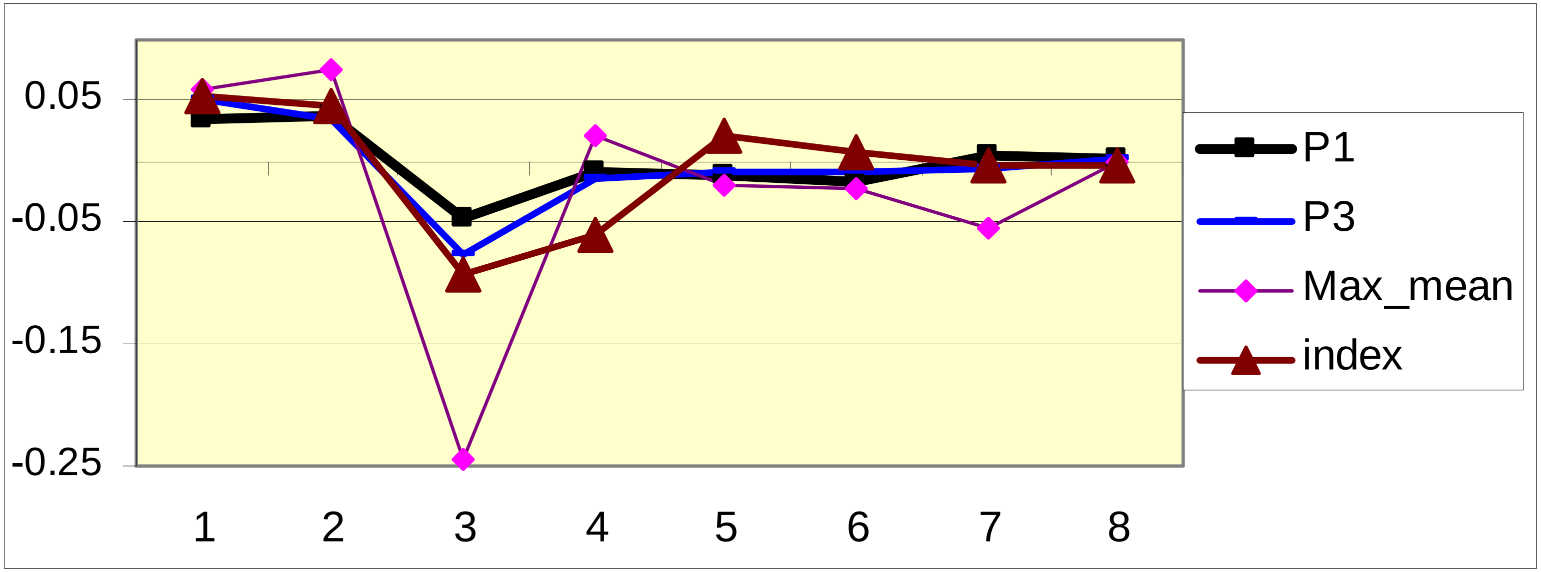
Computational results

Cases 1-3 : Summary of out-of-sample results



The performance of the Hang-Seng index, P1, P3 and Max-mean for the next 8 time periods following the date of selection (January - August 2004)

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Computational results

Comparison with mean-risk models

Purpose: select a portfolio whose return distribution has:

- High expected return
- Satisfactory left tail

Level of expected return: d_1=0.008726

- Mean -variance model $\rightarrow$ portfolio P_{var}
- Mean -0.95CVaR model $\rightarrow$ portfolio P_{CVaR}

Both: unsatisfactory left tail

Use our model model to improve on the left tail, maintaining a high mean level

Computational results

Comparison with mean-risk models

21 aspiration levels for:

- The worst 20 cumulative outcomes = their optimal values
- The expected value = d_1=0.008726 (aim on it!)

Run our model → portfolio P4

The return distribution of P4 has a slightly lower mean, but the improvement in the left tail is remarkable:

P_{var}	-0.210	-0.175	-0.152	-0.125	-0.123	-0.123	-0.111	-0.109	-0.101	-0.100	-0.098	-0.092	-0.089	-0.073	-0.072	-0.062	-0.060	-0.056
P_{CVaR}	-0.202	-0.175	-0.156	-0.130	-0.126	-0.122	-0.105	-0.105	-0.105	-0.100	-0.099	-0.094	-0.086	-0.085	-0.070	-0.056	-0.055	-0.053
P4	-0.174	-0.166	-0.147	-0.117	-0.116	-0.109	-0.093	-0.085	-0.079	-0.072	-0.069	-0.068	-0.067	-0.062	-0.059	-0.058	-0.056	-0.055

The left tail for return distributions of $P_{var,}$ P_{CVaR} and P4 as described by the worst 20 outcomes

Computational results

Comparison with mean-risk models

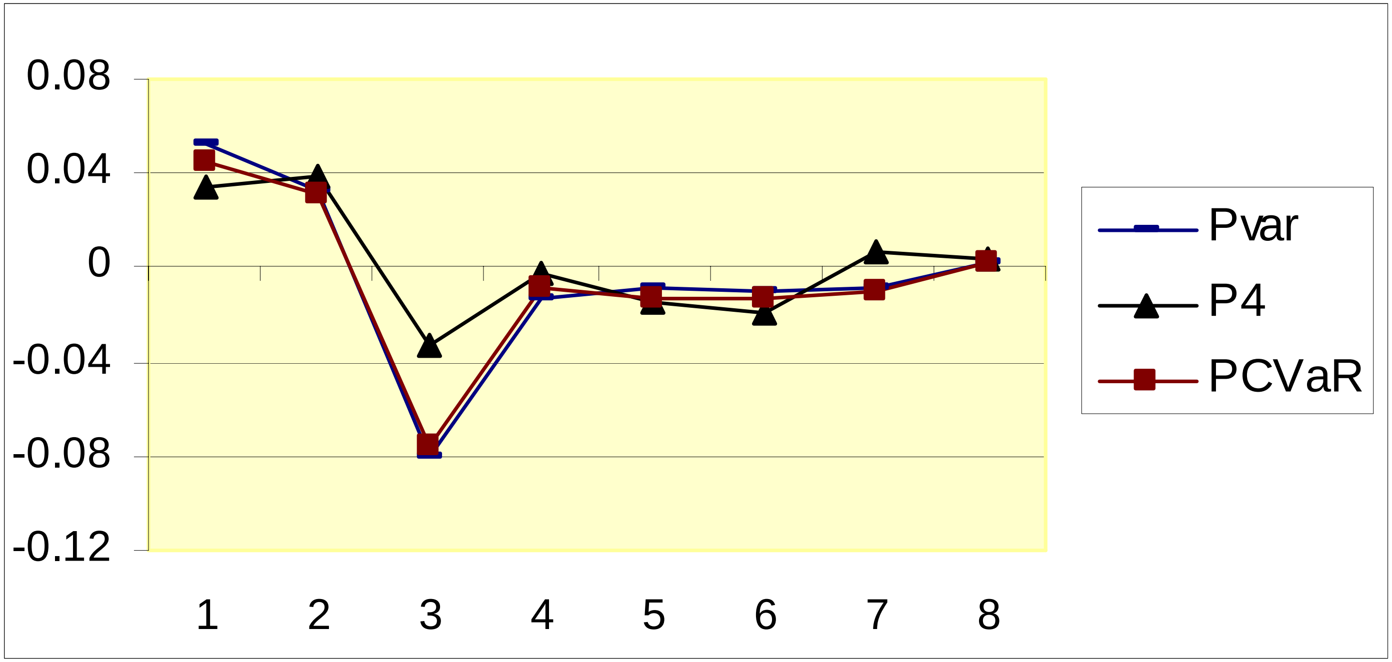
	P_{var}	P_{CVaR}	P4
Mean	0.008726	0.008726	0.007535
Median	0.008729	0.01108	0.010040
Standard Deviation	0.069247	0.069526	0.062459
Skewness	-0.166730	-0.14056	0.038810
Kurtosis	0.902741	0.827794	1.345610
Range	0.428051	0.417571	0.397412
Minimum	-0.210489	-0.2018	-0.173714
Maximum	0.217563	0.215768	0.223699

In-sample parameters for the return distribution of P_{var}, P_{CVaR} and P4

The return distribution of P4 has a slightly lower mean and median; the other in-sample parameters are better.

Computational results

Comparison with mean-risk models



The out-of-sample performance of P_{var}, P_{CVaR} and P4 for the next 8 time periods following the date of selection (January 2004-August 2004).

Portfolio P4 has in general better performance, avoiding extreme low values.

5. Concluding remarks

- The proposed model selects a meaningful solution using a computationally tractable procedure
- The distribution of the portfolio return can be shaped and “crafted” to a desirable form (in as much as this is possible).
- The solution is selected in an interactive way.
- The model can be used as enhanced indexation.
- The model can be used as a test for second order stochastic dominance of a given distribution.

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