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HMM based scenario generation for an investment optimization problem

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HMM based scenario generation for an investment optimization problem

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Abstract

We formulate an investment decision problem in a CVaR framework. An investor has to decide how much of his budget to invest into gold spots, FTSE 100 stocks or a bank account. Scenarios for this decision problem are generated based on a hidden Markov model filtering approach both in an independent as well as a correlated setting. Historical data sets for gold spots and the FTSE 100 are filtered to calculate optimal parameter estimates within the hidden Markov model. Scenarios for both time series are generated based on these optimal parameter estimates. Our findings show that the solutions obtained with independent generated scenarios are more stable than those found within a vector observation setting. This supports an independency of the two data sets considered, gold spots and the FTSE 100 index. The generated scenario sets are able to give stable solution for the considered decision problem with and without weight constraints. Furthermore a restated optimization problem focusing on maximising daily return subject to a given level of risk is solved with different scenario sets. The HMM parameter filtering method generates scenarios which lead to stable solutions for various investment optimization problems.

1 Introduction

Stochastic programming plays a very important role for portfolio optimization. The decision maker has a variety of choices where to invest money to achieve an optimal combination of risk and return. The outcome of the investment is uncertain, therefore techniques are developed to generate scenarios with numerous possible future returns, which try to give a realistic outlook into future performances. The problem formulation for stochastic optimization problems relies on different scenarios which need to be generated so the problem can be efficiently solved by stochastic programming algorithms. An overview of financial optimization problems can be found in Mulvey [7].

Various models for optimal portfolio selection have been proposed, starting with the popular Mean-Variance approach by Markowitz [6]. More recently portfolio optimization problems include more sophisticated risk measures. A popular risk measure in industry at the moment is Value-at-Risk (VaR), which indicates the maximum amount to be lost at a particular confidence level. As pointed out by Artzner et al. [2], one drawback of this measure in the application to stochastic optimization problems is that it is non-smooth and non-convex and can therefore lead to multiple local extrema. On the other hand, the conditional Value-at-Risk (CVaR), also known as Mean Excess Loss or Mean shortfall, is defined as the conditional expectation of portfolio returns below the VaR return and is a coherent risk measure (see Pflug [8]). Rockafellar and Uryasev [9] as well as Krokhmal et al. [4] developed CVaR models for portfolio optimization, one is to minimize risk with a given minimum level of portfolio return, the other is to maximise return with a given maximum level of risk. Krokhmal et al. [4] show in their article, that both formulations lead to the same efficient frontier of the portfolio under the CVaR framework. We apply both model formulations in our decision problem, the investor can therefore choose if he wants to fix the risk or return level in the optimization.

One important issue for various portfolio optimization problems is the scenario generation for underlying variables. Different approaches have been used to generate scenarios in

CVaR decision models. Andersson et al. [1] use Monte Carlo simulation for a credit risk optimization problem, Topaloglou et al. [11] generate scenarios with a principal component analysis for asset allocation in a CVaR framework. In our work we develop scenario generations based on a Hidden Markov model (HMM). Over the last years, HMM became increasingly popular for modelling financial time series. One main idea behind these models is that some information on the market is hidden in noisy observations. In our case these noisy observations are spots and indeces movements. The underlying hidden information can symbolize different stages of a business cycle, namely, expansion, peak, recession, trough and recovery, which influence the price movement. We generate scenarios based on optimal parameter estimations in the HMM. The parameter estimation filtering approach was pioneered by Elliott et al. [3]. Information of the Markov chain is filtered out of the observation process and optimal parameter estimates are derived with a reference probability measure technique. Once optimal parameters are estimated using historical information of the prices, we generate scenario paths evolving according to the parameter estimates and the filtered transition probability of the underlying Markov chain.

The report is organized as follows: In Section 2 we describe the investment problem and formulate the optimization problem. The scenario generation is derived in Section 3 for independent as well as vector observation processes. A step-by-step description of the scenario generator is given which can be applied to a variety of financial time series. Section 4 shows the results of the decision problem solved with different sets of scenarios, an extension to other problem formulations is given in Section 5. Our main findings show that the scenario generated is stable for various decision problem formulations. Solutions are found for both CVaR models, one with given minimal return and the other with given maximal risk. The conclusions are described in Section 6.

2 Investment problem

We consider a portfolio optimization problem. An investor faces the problem of finding the optimal ratio between investing in gold spots, FTSE100 stocks or putting his money into a bank account. We consider a one time-step problem, a decision has to be made concerning the percentages of a budget invested in gold, stocks or bonds. An investment in gold is traditionally seen as a long-term investment that can play a role in hedging against inflation and political or economic problems. Like other commodities, gold is traded on spot and future markets, but it has specific characteristics which are not common within commodity markets. Because of its role as a global currency, prices on spot markets are globally. Gold supply and demand does not depend on seasonality, it has a low risk of supply interruption and low storage and insurance costs. Furthermore gold has no risk of spoilage and the consumption level relative to inventory is low. Unlike price processes for other commodities, a convenience yield is not included in the price process due to the distinctive gold features. The performances of the spot prices and the FTSE 100 index are estimated under different scenarios, which are generated with parameters estimated by a hidden Markov model filtering method.

The decision in our model is based on calculating the conditional value at risk (CVaR) for the portfolio (see [10] for more details). For a given minimal portfolio return the portfolio with the lowest CVaR is chosen by the optimization algorithm. We follow the definition by Krokhmal et al. [4]. Let x be a portfolio from the set of available portfolios X and $f(x, y)$ the loss associated with the decision vector x, where y are uncertainties that can affect the loss. The probability of $f(x, y)$ not exceeding a threshold ζ is given by $\Psi(x, \zeta) = \int_{f(x,y)\leq\zeta} p(y)dy$, where $p(y)$ denotes the density of y. The $\alpha-$VaR and $\alpha-$CVaR for any specified probability level α in $(0, 1)$ is denoted by $\zeta_\alpha(x)$ and $\phi_\alpha(x)$ respectively and defined as

$$\zeta_\alpha(x) = \min\{\zeta \in \mathbb{R} : \Psi(x, \zeta) \geq \alpha\} \quad (1)$$

$$\phi_\alpha(x) = (1 - \alpha)^{-1} \int_{f(x,y)\geq\zeta_\alpha(x)} f(x, y)p(y)dy \quad (2)$$

where $\phi_\alpha(x)$ is the conditional expectation of the loss associated with the portfolio relative to that loss being $\zeta_\alpha(x)$ or greater. Define a function $F_\alpha(x, \zeta)$ on $X \times \mathbb{R}$ by

$$F_\alpha(x, \zeta) = \zeta + (1 - \alpha)^{-1} \int_{y\in\mathbb{R}^n} \left[f(x, y) - \zeta\right]^+ p(y)d(y) \quad (3)$$

with $[t]^+ = \max\{t, 0\}$. As a function of ζ, $F_\alpha(x,\zeta)$ is convex and continuously differentiable. The $\alpha-$CVaR can then be determined from

$$\phi_\alpha(x) = \min_{\zeta \in \mathbb{R}} F_\alpha(x,\zeta). \tag{4}$$

Minimizing the $\alpha-$CVaR is equivalent to minimizing $F_\alpha(x,\zeta)$ over all $(x,\zeta) \in X \times \mathbb{R}$ (see Krokhmal et al. [4], Theorem 2), so

$$\min_{x \in X} \phi_\alpha(x) = \min_{(x,\zeta) \in X \times \mathbb{R}} F_\alpha(x,\zeta). \tag{5}$$

Discretizing the problem formulation leads to the approximation

$$\tilde{F}_\alpha(x,\zeta) = \zeta + (1-\alpha)^{-1} \sum_{s=1}^{S} Prob_s [f(x,y_s) - \zeta]^+ \tag{6}$$

where $Prob_s$ are the probabilities of scenario y_s.

The defining elements of our optimization problem are

- **Indices**
  $Scen = 1, ..., S$: scenarios

- **Data**
  $no\_scenarios$: number of scenarios generated for spot and stock returns
  $ReturnSpot_s$: gold spot price return for scenario s
  $ReturnFTSE_s$: FTSE100 return for scenario s
  $Prob_s := \frac{1}{no\_scenarios}$ probability of scenario s
  $ReturnBond$: Return on bank account
  $MReturnSpot$: Mean gold price return over all scenarios s
  $MReturnFTSE$: Mean FTSE100 return over all scenarios s
  β : Confidence level for CvaR
  $MinPortRet$: minimal required level of expected portfolio return

- **Decisions**
  $BuySpots$: percentage of budget invested in gold spots
  $BuyFTSE$: percentage of budget invested in FTSE100 stocks
  $BuyBond$: percentage of budget invested in bank account
  $negdev_s$: difference between VaR and portfolio return in scenario s
  $alpha$: level of VaR
  Z : objective function

- **Objective:** Minimize Cvar

$$Z = \alpha + \frac{1}{\beta} * \sum_{s \in Scen} Prob_s * negdev(s)$$

- **subject to**

  1. Weight constraint:
     $\text{BuySpots} + \text{BuyFTSE} + \text{BuyBond} = 1$
  2. Minimal portfolio return:
     $\text{BuySpots}*\text{MReturnSpots}+\text{BuyFTSE}*\text{MReturnFTSE}+\text{BuyBond}*\text{ReturnBond} \geq MinPortRet$
  3. CVaR constraint:
     $\text{BuySpots} * \text{ReturnSpot}_s + \text{BuyFTSE} * \text{ReturnFTSE}_s + \text{BuyBond} * \text{ReturnBond} + \alpha + negdev_s \geq 0 \ \ \forall s \in \ Scen$
  4. Positivity constraints:
     $negdev \geq 0$
     $\text{BuySpots} \geq 0, \ \text{BuyFTSE} \geq 0, \ \text{BuyBond} \geq 0$

3 Scenario generation

The price of gold and the FTSE 100 index, as other stock prices, can be modelled with a geometric Brownian motion. Since gold spot prices unlike other commodity prices do not show seasonality etc. a convenience yield is not included in the price process and it can therefore also be modelled by a geometric Brownian motion. The parameter estimation and forecast we calculate here demands for a discretized geometric Brownian motion as an observation process. The scenarios are generated under a hidden Markov model framework. In this setting, we assume to have a noisy observation process with a hidden pattern. This pattern is represented by a Markov chain in discrete time. Different states of the Markov chain represent different states of the economy. For example, if the economy is booming the Markov chain is in another state then in a recessive economic state. Since we want to allow our model to switch between these different economic regimes the parameters in our observation processes are governed by this underlying Markov chain, which is not directly observable. For our scenario generation we assume that the log returns of both price processes have the following dynamics in discrete time

$$y_{k+1} = f(\mathbf{x}_k) + \sigma(\mathbf{x}_k) z_{k+1}. \tag{7}$$

The parameters f and σ are governed by the Markov chain $\mathbf{x}$ in discrete time and are therefore able to switch between different regimes. The $z_k's$ are a sequence of independent, identically distributed (IID) standard normal random variables independent of $\mathbf{x}$, they form the white noise term of the observation process. We denote the scalar product with $< \cdot, \cdot >$, the parameters are of the form $\mathbf{f} = (f_1, f_2, \dots, f_n)^\top$ and $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \dots, \sigma_n)^\top$ such that $f(\mathbf{x}_k) = \langle \mathbf{f}, \mathbf{x}_k \rangle$ and $\sigma(\mathbf{x}_k) = \langle \boldsymbol{\sigma}, \mathbf{x}_k \rangle$ where σ_i's are all positive for every $1 \leq i \leq n$. The observation at time $k+1$ depends on the state of $\mathbf{x}$ at time k. This is a one-step delay model and is reasonable as y may not react to $\mathbf{x}$ immediately.

Under the real world probability measure P, the Markov chain $\mathbf{x}$ has the dynamics

$$\mathbf{x}_{k+1} = \mathbf{\Pi} \mathbf{x}_k + \mathbf{v}_{k+1} \tag{8}$$

where $\mathbf{v}_{k+1}$ is a martingale increment and $\Pi = (\pi_{ji})$ is the transition probability matrix with $\pi_{ji} = P(\mathbf{x}_{k+1} = \mathbf{e}_j | \mathbf{x}_k = \mathbf{e}_i)$.

We developed filters for the Markov chain and related quantities and derive recursive optimal parameter estimation for the regime-switching processes. The jump process of the

Markov chain, namely the number of times the Markov chain jumps from state i to state j up to a given time k is denoted by $J_k^{(ji)}$.

$$J_k^{(ji)} = \sum_{l=1}^{k} \langle \mathbf{x}_{l-1}, \mathbf{e}_i \rangle \langle \mathbf{x}_l, \mathbf{e}_j \rangle \tag{9}$$

The occupation time process which indicates how long the Markov chain stays in state i up to time k is denoted by $O_k^{(i)}$.

$$O_k^{(i)} = \sum_{l=1}^{k} \langle \mathbf{x}_{l-1}, \mathbf{e}_i \rangle \tag{10}$$

Furthermore we need an auxiliary process $T_k^{(i)}$ for filtering the Markov chain.

$$T_k^{(i)}(g) = \sum_{l=1}^{k} \langle \mathbf{x}_{l-1}, \mathbf{e}_i \rangle g(y_l) \tag{11}$$

where g is a function that will take the form $g(y) = y$ or $g(y) = y^2$. (12)

These processes are filtered out of our observation process y under a reference probability measure following a filtering method by Elliott, Aggoun and Moore [3]. Write $\mathbf{D}$ for the diagonal matrix whose ith entry on the diagonal is

$$\frac{\phi\left(\frac{y_{k+1}-f_i}{\sigma_i}\right)}{\sigma_i \phi(y_{k+1})}. \tag{13}$$

The filters for the processes are then derived as

$$\begin{aligned} \gamma(J^{(ji)}\mathbf{x})_l &= \mathbf{\Pi D}(y_l)\gamma(J^{(ji)}\mathbf{x})_{l-1} \\ &\quad + \left\langle \boldsymbol{\xi}_{l-1}, \mathbf{e}_i \right\rangle \frac{\phi\left(\sigma_i^{-1}(y_l - f_i)\right)}{\sigma_i \phi(y_l)} \pi_{ji} \mathbf{e}_j, \quad (14) \\ \gamma(O^{(i)}\mathbf{x})_l &= \mathbf{\Pi D}(y_l)\gamma(O^{(i)}\mathbf{x})_{l-1} \\ &\quad + \left\langle \boldsymbol{\xi}_{l-1}, \mathbf{e}_i \right\rangle \frac{\phi\left(\sigma_i^{-1}(y_l - f_i)\right)}{\sigma_i \phi(y_l)} \mathbf{\Pi e}_i \quad (15) \\ \gamma(T^{(i)}(g)\mathbf{x})_l &= \mathbf{\Pi D}(y_l)\gamma(T^{(i)}(g)\mathbf{x})_{l-1} \\ &\quad + \left\langle \boldsymbol{\xi}_{l-1}, \mathbf{e}_i \right\rangle \frac{\phi(\sigma_i^{-1}(y_l - f_i))}{\sigma_i \phi(y_l)} g(y_l) \mathbf{\Pi e}_i. \quad (16) \end{aligned}$$

with $\boldsymbol{\xi}_k := \widetilde{E}[\Lambda_k \mathbf{x}_k | \mathcal{Y}_k]$ and $\boldsymbol{\xi}_{k+1} = \mathbf{\Pi D} \boldsymbol{\xi}_k$. More details and proofs for these filter derivations can be found in [5].

The derived filters for the processes are used for the recursive parameter estimation formulas. The recursive formualas for the model parameters are calculated with the Expectation-Maximization (EM)- algorithm. The maximum likelihood estimation (MLE) for the parameters within the EM-algorithm makes use of the derived adaptive filters of the observation process. In the calculation of the MLE, the filters substitute terms involving the observation process. The following recursive optimal parameter estimates for the transition probabilities π_{ji}, the mean f_i and the variance σ_i of the observation process are derived:

$$\widehat{\pi}_{ji} = \frac{\widehat{J}_k^{(ji)}}{\widehat{O}_k^{(i)}} = \frac{\gamma(J^{(ji)})_k}{\gamma(O^{(i)})_k}, \tag{17}$$

$$\widehat{f}_i = \frac{\widehat{T}_k^{(i)}}{\widehat{O}_k^{(i)}} = \frac{\gamma(T^{(i)}(y))_k}{\gamma(O^{(i)})_k} \tag{18}$$

and

$$\widehat{\sigma}_i = \sqrt{\frac{\widehat{T}_k^{(i)}(y^2) - 2\widehat{f}_i \widehat{T}_k^{(i)}(y) + \widehat{f}_i^2 \widehat{O}_k^{(i)}}{\widehat{O}_k^{(i)}}}. \tag{19}$$

With these optimal parameter estimates we are able to calibrate a one-step ahead forecast for the data series. A forecast of gold prices follows the actual data very closely (see [5] for a case study on forecasting gold spot prices).

We consider a three-state Markov chain for our scenario generation. For most actual data series three states are sufficient to capture different states of the economy without overfitting the model. The input for the scenario generator are historical data for the time series, a period of one year is sufficient to estimate optimal parameters.

The scenario generation takes the following steps:

1. Estimate optimal parameters for the time series with an underlying three-state Markov chain.

2. Estimate the optimal transition probability matrix for the considered time series.

3. Create random normal distributed variables, which are used for generating the white noise part.

4. Generate scenarios for the next time step:
   a) the Markov chain for the next time step is calculated with its expectation conditioned on the general filtration $\mathcal{Y}_k$. We therefore have $E[\mathbf{x}_{k+1}|\mathcal{Y}_k] = \Pi\hat{\mathbf{x}_k}$ with $\hat{\mathbf{x}_k} = E[\mathbf{x}_k|\mathcal{Y}_k]$.

   b) the parameter values for the next time step are calculated as the scalar product between the expected Markov chain $\mathbf{x}_{k+1}$ and the estimated optimal parameters,

   $$f_{scen} =< f, \mathbf{x}_{k+1} > \text{ and } \sigma_{scen} =< \sigma, \mathbf{x}_{k+1} > .$$

   c) the scenarios for the prices in the next time step are created with the discretized version of the geometric Brownian motion using the parameters stated above

   $$S_{scen} = C(k) * \exp(f_{scen} + \sigma_{scen} * w_{scen}),$$

   where $C(k)$ denotes the last actual data point of the observation process and w denotes the IID standard normal distributed random variable independent of the Markov chain for Scenario $scen$.

   d) the required log returns of the data series are then calculated for each scenario using the estimated prices calculated in the different scenarios above.

In the following sections we consider two different frameworks for the scenario generations. In the first setting both time series are assumed to be independent, the optimal parameters for each process are derived in separate algorithms. The second setting assumes a dependency between the time series. Both observation processes are governed by the same Markov chain and are therefore estimated as vector observations.

3.1 Independent observations

Our underlying historical data series are time series for daily prices from gold spots and the FTSE 100 index over a 1-year period. The algorithm for the optimal parameter estimates runs 24 times on these data sets. First the time series are assumed to be independent, therefore both data sets follow independent Markov chains. For each data set the parameters are updated when new information arrives after batches of 10 data points. With this self-tuning algorithm for the regime-switching observation process optimal parameter estimates are calculated and the transition probability matrix of the Markov chain $\mathbf{x}$ is estimated for the gold price and FTSE 100 time series. With these optimal parameter estimates we generate scenarios for both price processes assuming that the unknown noise for the next time step follows a Brownian motion. Figure 1 and 2 depict the optimal parameter estimates for the gold spot prices and for FTSE 100, respectively. Scenarios are generated starting from the last data point of the two return series. These scenarios are then used for the investment decision.

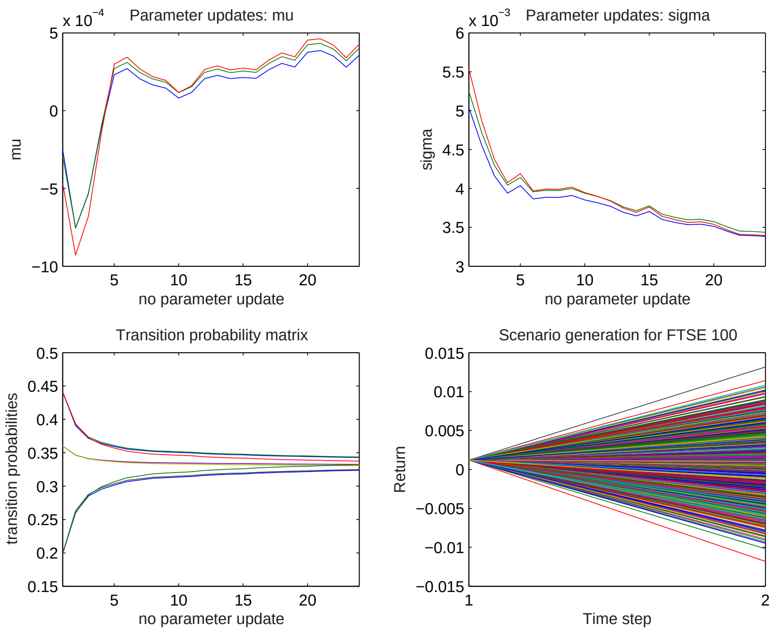

Figure 1: Parameter estimation and scenario generation for FTSE 100

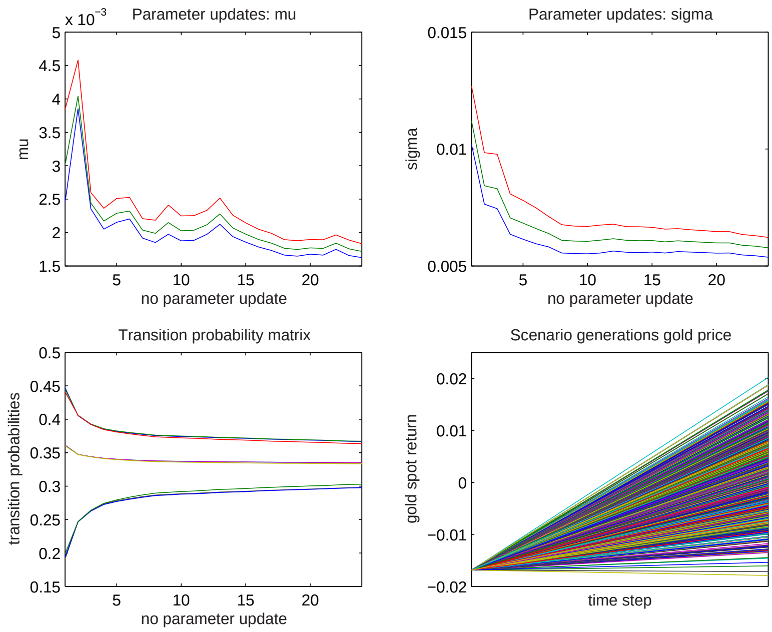

Figure 2: Parameter estimation and scenario generation for gold spots

3.2 Vector observations

The aim of this section is to perform scenario generations under the assumption that the time series of FTSE 100 and gold spots are correlated. Due to the nature of our scenario generator, we do not include a correlation factor between the two observation processes but we assume that both time series are governed by the same hidden Markov chain. Under this framework we estimate the optimal parameters of both observation processes simultaneously, the processes are assumed to be vector observations. Therefore the underlying Markov chain is filtered out from both processes, all parameters depend on the same Markov chain representing economic states which occurred over the considered time interval. A thorough discussion on the filter derivation and optimal parameter estimation for vector observations can be found in Elliott [3].

When the parameters are derived for both time series, we perform the same scenario generation as discussed above. The one-step ahead scenarios are generated with the derived parameter values. Uncertainty is added by standard normal IID random variables, which are assumed to be independent for the two processes. The figure below shows the parameter estimation of the observation processes in this vector observation case. Possible scenarios for both time series are generated.

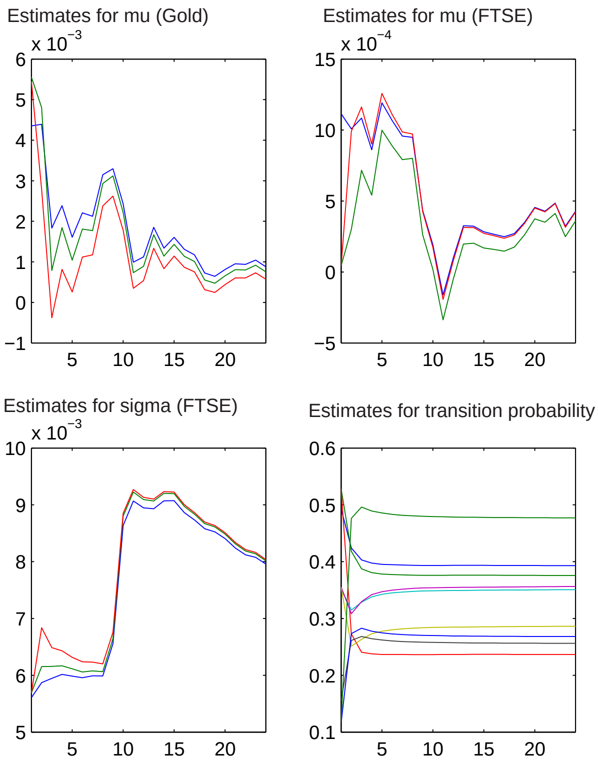

Figure 3: Simultaneous parameter estimation for observation process of gold spots and FTSE 100 returns

4 Investment decisions

This section describes investment decisions taken with the independent as well as the correlated scenario generations. First we examine the solutions in the independent case for

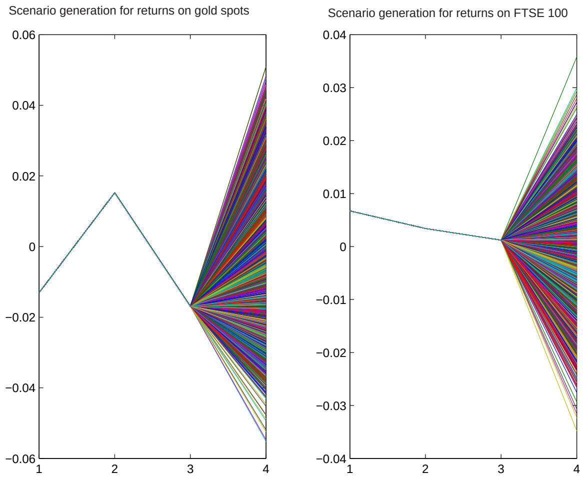

Figure 4: Scenario generation for returns on gold spots and FTSE 100

different values of required minimal portfolio return as well as for ten different generated scenario sets. This is also done for the correlated case, the resulting solutions are compared with respect to the stability of the generated scenarios. The optimization problem is solved in AMPL with the FortMP solver for a one-step time setting.

4.1 Scenario Generation with independent observations

Solutions for the investment problem are first obtained based on the independent generated scenarios. The minimal portfolio return d is constant, the daily return on the bank account is assumed to be 0.0001. The confidence level of CvaR, β is set to 1%. Both FTSE 100 and gold spot prices have n generated scenarios, which occur with equal probability.

The first comparison gives us the objective function and optimal portfolio weights for two different minimal portfolio returns d for 2470 generated scenarios. For the first optimization d is set to 0.0004. The following solution for the objective function and the percentages to be invested in gold, FTSE 100 stocks and the bank account is obtained:

Objective function:	CVaR:	0.00220126665512158
Decision variables:	BuyFTSE:	0.0710667
	BuySpots:	0.155712
	BuyBond:	0.773221

In this case, the optimal solution is to invest the largest amount of money (77.3%) in the bank account, followed by 15.6% to invest in gold spots and 7.1% to invest in FTSE 100 stocks. If the minimum required portfolio return is increased to $d = 0.0005$, the optimal solution gives us a higher objective function. The objective and percentages invested in the different possibilities are then

Objective function:	CVaR:	0.0029683555401621084
Decision variables:	BuyFTSE:	0.0947556
	BuySpots:	0.207617
	BuyBond:	0.697628

The higher percentages for investing in stocks and gold spots is due to the fact that a higher minimal return is needed and therefore more risk is taken. The lower CVaR value for the lower minimal return $d = 0.0004$ reflects the lower risk of investing in a bank account.

An analysis of the scenario generation is performed on different generated scenario sets. We generate 10 sets with 5000 scenarios each and calculate the optimal solution for these sets. The minimal required level of daily portfolio return is set to $d = 0.0005$. The resulting values for the objective function as well as the the optimal percentages invested in FTSE 100 stocks, gold or bonds can be seen in the table below. All values of the objective function lie between 0.002844 and 0.003325 with $mean = 0.0031$ and $var = 2.7070e - 008$.

Scenario set		1	2	3	4	5	6	7	8	9	10
Objective function		0.003325	0.00321	0.002955	0.003199	0.003222	0.003035	0.003004	0.002844	0.002927	0.002898
Percentages invested in:	FTSE	9.77	17.64	9.14	16.88	15.27	12.03	15.84	10.38	13.63	8.81
	Gold	24.37	20.71	22.26	21.8	22.76	23.08	21.08	21.44	21.8	22.38
	Bond	65.96	61.65	68.6	61.32	61.97	64.89	63.08	68.18	64.57	68.81

Table 1: Results for 10 scenario sets with independent observations

4.2 Scenario Generation for vector observations

In this section we perform scenario generations under the assumption, that both FTSE 100 index and gold spots follow the same Markov chain. Ten sets of 10,000 scenarios are generated and the optimal solution is calculated for each of these scenario sets. The minimal required portfolio return is set to $d = 0.0005$ and the CVaR confidence level is $\beta = 0.01$. The table below shows the resulting values for the objective function as well as the corresponding optimal portfolio weights. The value of the objective function lies between 0.015058 and 0.027259 with a mean of 0.0193 and variance of $1.4093e - 005$.

Scenario set		1	2	3	4	5	6	7	8	9	10
Objective function		0.018185	0.015058	0.015768	0.021392	0.027259	0.016276	0.023035	0.019834	0.017423	0.0185903
Percentages invested in:	FTSE	42.47	46.13	40.58	99.47	30.78	58.77	43.06	63.75	47.06	63.1
	Gold	39.15	28.83	35.16	0.53	69.22	27.22	56.94	36.25	37.52	36.9
	Bond	18.38	25.04	24.26	0	0	14.01	0	0	15.42	0

Table 2: Results for 10 scenario sets with vector observations

The solutions obtained with these generated scenarios are less stable than those obtained with the independent scenario generator. They variation is a lot higher, the highest value 0.027259 is nearly twice as high as the lowest objective 0.015058. The investment percentages for the three different categories vary largely between different scenario sets. In set 2, 25.04% shall be invested into the bank account compared to 0% in scenarios 4, 5, 7, 8 and 10. Compared to the objective function of the previous scenario generations, the values of the objective function are more than six times higher in the correlated setting, the variance is still low, but also significantly higher than the variance obtained by the other scenarios generated. These deviations of the objective function over different scenario sets leads to the conclusion, that our model requires independent generated scenarios. Economically this can be explained by the largely uncorrelated price of gold to any other stock. Due to its characteristics as a stable long-term investment and a global currency, gold prices are highly likely to be independent of stock prices or indeces. This is supported by stable optimal solutions calculated with independent generated scenarios.

5 Extended investment problems

A further analysis of generated scenario sets under different problem settings is described in this section. One optimization problem with maximal weight constraints is examined as well as a problem formulation for maximising the portfolio return for a given CVaR value.

5.1 Investment problem with weight constraints

In this section we modify our original investment problem and add constraints on the weights invested in the different assets. All weights now have an upper bound of 50%, no more than half of the budget is allowed to be invested in either gold spots, FTSE 100 stocks or bonds. Taking into account the results from the previous section, we solve this investment problem with scenario sets generated by the independent scenario generator. We solve the problem with five sets containing 5,000 and five sets containing 20,000 scenarios. The objective functions obtained are again stable over the ten scenario sets with different number of scenarios. As expected from the previous results, a weight constraint of 50% cuts down the budget part invested in bonds from around 65% to exactly 50%. The difference is mostly invested into FTSE 100 stocks, resulting in a mean solution over ten scenarios of roughly 31% invested in FTSE 100 stocks, 19% invested in gold spots and 50% invested in the bank account. The average objective function over all generated scenario sets is 0.003538409 with

Scenario set		5,000 generated scenarios					20,000 generated scenarios				
		1	2	3	4	5	1	2	3	4	5
Objective function		0.003676	0.003541	0.003575	0.003459	0.00354	0.003522	0.003569	0.003517	0.003465	0.00352
Percentages invested	FTSE	29.38	32.67	32.6	31.57	30	31.84	30.55	31.87	32.04	30.05
	Gold	20.62	17.33	17.4	18.43	20	18.16	19.45	18.13	17.96	19.95
	Bond	50	50	50	50	50	0.5	0.5	0.5	0.5	0.5

Table 3: Investment problem with weight constraints, results for 10 scenario sets

a very low variance of $3.77072e-09$. The scenario generator gives sets which lead to stable solution of the investment problem with weight constraints.

5.2 Reformulated optimization problem

After the analysis of the original CVar optimization problem, namely minimizing the CVar of the portfolio under a given minimal required level of portfolio return, we reformulate the problem now to be able to optimize the return. Our objective function is now to minimize the negative expected return over all scenarios, we set an upper bound for CVar. This optimization formulation reflects the decision problem to maximise the return under a given risk level. We adapt the framework derived by Krokhmal et.al. [4]. The optimization problem is given by the objective function

$$\begin{aligned}
\text{minimize} \quad & negativeReturn : \\
& -(BuyFTSE * MReturnFTSE + BuySpots * MReturnSpot \\
& \quad + BuyBond * ReturnBond) \\
\text{subject to} \quad & \text{CVaR:} \\
& \alpha + \frac{1}{no_scenarios * (1-\beta)} * \sum_{s\ in\ Scen} (negdev[s]) \ \leq \ w \ .
\end{aligned}$$

First, the optimization is done on one scenario set with 5000 scenarios. An efficient frontier can be calculated by setting varying values of w, the percentage of the portfolio which is allowed for risk exposure. The efficient frontier is plotted in Figure 3 for three $\beta-$values, $\beta = 0.9, \beta = 0.95$ and $\beta = 0.99$.

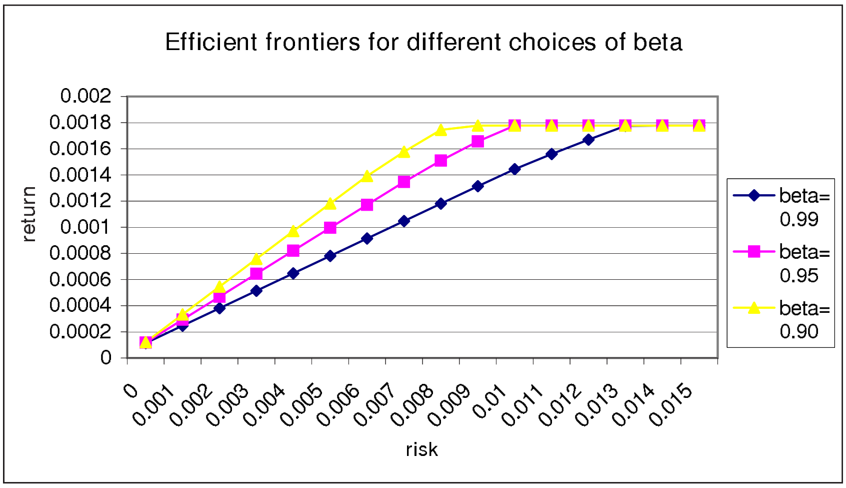

Figure 5: Efficient frontier for portfolio selection

Now we generate again scenario sets with the generator for independent observations. We generate five sets with 5,000 scenarios and five with 10,000 scenarios to test the in-sample stability of the scenario generator. The table below shows the results for maximising the portfolio return by a given level of $\beta = 0.99$. and $w = 0.005$. We also include the weight constraint $<= 50\%$ in this optimization problem.

Scenario set		5000 generated scenarios					10000 generated scenarios				
		1	2	3	4	5	1	2	3	4	5
Objective function		-0.000732	-0.000802	-0.000771	-0.00076	-0.000672	-0.000715	-0.000709	-0.00076	-0.000721	-0.000741
Percentages invested in:	FTSE	19.51	27.38	22.81	19.49	20.65	15.86	16.79	17.21	24.84	21.32
	Gold	35.03	34.34	34.81	35.57	34.01	35.72	34.8	36.47	33.16	34.18
	Bond	45.46	38.28	42.38	44.94	45.34	48.42	48.41	46.32	42	44.51

Table 4: Minimized negative return and weights for different scenario sets

The solutions for this problem are quite stable throughout the scenario sets. The mean value of the objective function is -0.000738382 with corresponding mean percentages of 20.59% to invest in FTSE 100 stocks, 34.81% to invest in gold and 44.60% to put into a bank account. The variance of the objective function is very low, var $= 1.35182et - 09$.

6 Conclusion

We have solved an optimal investment problem in a CVaR framework based on scenarios generated with a hidden Markov model. The HMM setting enables data series' parameters to switch between different economic regimes. In our case, the spot price of gold and the FTSE 100 index was modelled in a HMM framework. Based on historical daily data, optimal parameter estimates are calculated which are then used for the generation of scenarios. Independently generated scenarios lead to stable solutions of the optimization problem, here both time series were assumed to have one underlying Markov chain. Scenarios generated on the basis of vector observations, meaning that both time series follow one underlying Markov chain and are therefore dependent on the same economic triggers, lead to less stable results, which supports the fact that the evolution of gold spots is widely independent of the FTSE 100. Generated scenarios were also tested on the investment problem including weight constraints on the percentages invested in different assets. Furthermore the CVaR optimization problem was reformulated so that the daily return is maximised subject to a given level of risk. This problem was solved with ten different scenario sets having 5,000 or 10,000 scenarios each. Again, the generated scenarios lead to stable solutions. The HMM scenario generator is therefore able to provide scenario sets for a variety of investment problems. The focus of future research will be to extend and test the scenario generator for multiple time step optimization problems.

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