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Aggregation of Market Risk and Credit Risk

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Aggregation of Market Risk and Credit Risk
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Oliwia Kozlowska

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Introduction

The original purpose of the report was to develop a risk aggregation methodology that would establish a link between market risk and credit risk at the level of the whole organisation[1]. The developed methodology aims at aggregating market risk and credit risk measurement, so that the joint economic capital[2] can be calculated. The link between the two risks was required to reach beyond the information conveyed by the correlation coefficients, and for this reason it was decided to employ the copula theory. At the same time, the developed methodology was required to ensure the analytical tractability of the large number of risk factors affecting the organisation as a whole.

For banks, the need for providing information on their overall risk profile is reflected by Basel II requirements[3]. The requirement for comprehensive assessment of risks is explicitly stated in Pillar II ([BI05], p. 159 - 162). From there, recognition of the diversification effects across different risk types leads to the concept of an integrated risk measurement process. The same concept would apply to organisations operating on regulated markets.

Calculating joint economic capital is especially important while aggregating market and credit risk, because the diversification effect between these two risk types can be substantial. This, in turn, can lead to reduction in the overall economic capital, especially if compared to addition of economic capital calculated for each of the risk types separately - which corresponds to the case as if they were fully correlated. The evidence presented in this report shows that they are not, but - given how risky the situation can be if an organisation is adversely affected by both risks - it is acknowledged that the dependence between the two risk types should be determined with maximum accuracy. This is why the proposed dependence structure goes beyond correlation and relies on a more accurate dependence pattern defined by a copula.

[1]The risk aggregation research project was contracted by BP Oil International: BP Oil International Limited, a company registered in England and Wales with the company number 322365 and VAT Number GB 243 5105 93 and whose registered office is Chertsey Road, Sunbury on Thames, Middlesex, TW16 7BP.

[2]"Risk capital" is used equivalently to "Economic capital" - both meaning the capital, which a firm requires in order to cover the risks that it is exposed to.

[3]See Principle 1 of supervisory review: "Banks should have a process for assessing their overall capital adequacy in relation to their risk profile and a strategy for maintaining their capital levels" in [BI05].

The literature provides a number of concepts of aggregating market and credit risk, especially when pricing a single financial instrument incorporating both types of risk - e.g the bond pricing model introduced by in [Tur00].
Another attempt to establish a link between credit risk and the interest rate as a major market risk factor - in the context of their joint impact on a bank's balance sheet, has been made in [MD07]. A similar procedure, encompassing the macroeconomic state as a whole, has been employed in [AC04]. There, the bank's loan portfolio is linked to the state of the macroeconomic business cycle, summarised by the variable referred to as the "Dynamic Factor". Integrating the interest rate and the credit spread risk with credit portfolio models, with focus on the effect on the economic capital, was also approached in [Gru05].
A broader portfolio context for aggregated market and credit risk has been proposed in [II99]. The proposed model clearly identifies market risk factors and credit risk factors (drivers), defining a pattern of their joint evolution over time. The construction of the model derives from factor models - by splitting systemic credit factors (drivers) and idiosyncratic components associated with sectors of portfolio obligors, and proceeding from conditional to unconditional default probabilities. Further technical details, including Monte-Carlo techniques relevant for this approach can be found in [Kre01].
Following the logic of the Jarrow and Turnbull's model, as well as of the one by Iscoe, Kreinin and Rosen, in [MKB01] the Merton's firm value model is recalled, with an attempt to integrate credit, market and foreign exchange risk within the portfolio of a financial institution, as well as to evaluate their effect while stress-testing.
In this report the aggregation of market and credit risk is approached at the level of the whole organisation operating in an environment affected by both risks simultaneously. A similar mathematical concept has been recently proposed in [CG07] and [OK07] - both in the context of calculating economic capital as a joint risk figure representative for a bank's overall risk profile. These two articles, provided the motivation for the systematics of the risk aggregation methodologies included in this report.

The aggregation of market and credit risks can be done at different levels of detail. The ideal would be to know the joint behaviour of all market and credit risk factors that affect an organisation. This is the most detailed approach. Obviously the number of risk factors affecting the whole organisation is very large, leading straight to the high dimensionality problem. High dimensionality is also a major obstacle while determining the dependence between market risk and credit risk. This is why the focus of this report is on overall performance of the organisation in terms of its activity on the market (the source of market risk) and the portfolio of its obligors (the source of credit risk), regardless of the number of underlying individual risk factors. This can be understood as the aggregation at the level of overall market performance and overall credit performance. When dealing with overall performance in place of all individual risk factors some information is lost - but the case becomes computationally tractable. The aim of this report is to investigate how this approach is:

- accurate,
- useful for risk management,
- sensitive to adverse movements of individual risk factors.

Additional motivation is that there is another - even less detailed - aggregation level: risks can be aggregated not by variables, but by risk measures (e.g. VaRs). However, the accuracy and tractability might depend on the assumed distribution properties of the risk factors. An example of this kind of technique - applicable to normally distributed risk factors and delta-normal VaR as risk measure - can be found in [Jor], and is also recalled in [OK07].
The comparison among aggregation methodologies of market and credit risk, providing different level of detail on the organisation's overall risk profile, adds value to the concept proposed in [CG07] and [OK07]. Referring to the most detailed approach, which involves modelling all individual risk factors, allows us to evaluate how much information on the overall risk profile is preserved, as well as to develop techniques useful for implementation in a large-size organisation.

For a start, the comparison among risk aggregation methodologies forces to use the small-size model, for which all the methodologies are accessible. On one hand, the use of the small-size model means that drawing any conclusions, that would equivalently hold for the large-size organisation, needs special attention and detail. However, on the other hand - it allows us to focus on the model properties and the numerical results, going well beyond the very general statement of the problem.
The concept proposed in [CG07] and [OK07] concerns the economic capital, which is a Value at Risk-based measure. In this report the focus is on Value at Risk itself, as well as on Expected Shortfall being a more detailed risk measure. Nevertheless, the logic behind the construction of aggregate (global) variables summarising market risk and credit risk - and behind establishing the link between them - is the same. However, unlike the publications cited above, this report does not attempt to provide close-form solutions, but rather focuses on numerical results, illustrating how much information on the overall risk profile is preserved by each of the risk aggregation methodologies. This approach allows us to compromise on many of the assumptions regarding the properties of the individual risk factors underlying each of the aggregate (global) variables.
The range of this report goes as far as to investigate several dependence patterns between the aggregate (global) variables, as well as of the problem of building the historical data sample by varying exposures.
Summarising, the contribution of this report is threefold: First - to propose a risk aggregation methodology satisfying the criteria for implementation on a large scale, with a sufficient

level of mathematical accuracy in terms of defining the dependence structure between market risk and credit risk. Second, the properties of the proposed methodology are analysed vs. benchmark methodologies, i.e. vs. methodologies marking the extremes of mathematical accuracy and analytical tractability. Finally, the scale of possible savings in economic capital of an organisation is analysed.

The rest of the report is organised as follows: In the first chapter, the systematic of the risk aggregation methodologies is provided. Chapter two explains details of the small-size model, to which the proposed risk aggregation methodology is applied. Finally, in chapter three, the same methodology is applied to the large-size model, and its impact on risk measurement process is analysed.

Chapter 1

Methodology of aggregating market risk and credit risk

This chapter provides a systematic of risk aggregation methodologies. Following the original purpose of the report - to propose a risk aggregation methodology which is both mathematically accurate and applicable at the level of the whole organisation, the special focus is on two characteristics: first - the information on the risk profile that each of the methodologies provides, and second - the scale, on which it can be implemented.
In this context, the construction of the loss aggregation methodology is explained. Its main goal is to find the joint distribution of two random variables: total market loss encompassing the organisation's overall market activity (the source of market risk), and total credit loss encompassing the credit quality of the organisation's obligors (the source of credit risk).
For the purpose of this analysis, the bivariate case with total market loss and total credit loss as the two random variables is referred to as "loss aggregation". Loss aggregation is a compromise between two methodologies marking the extremes of mathematical accuracy and analytical tractability (benchmark methodologies): first, the methodology conveying maximum information on the risk profile but being far too complex for a large number of risk factors, which is herein referred to as "risk factor aggregation"; Second the methodology conveying minimum information on the risk profile but being analytically tractable even for a very large number of risk factors, referred to as "risk measure aggregation" or "VaR aggregation" - if Value at Risk (VaR) is the currently used risk measure. Loss aggregation is positioned as medium, both in terms of the amount of information on the risk profile it provides, and its analytical tractability.
The rest of the chapter explains the general construction and the level of information detail on the risk profile, provided by each of the methodological approaches - with emphasis on loss aggregation.

1.1 Levels of risk aggregation

We consider three levels of risk aggregation, directly related to the levels of information detail on the organisation's risk profile.

1.1.1 The ideal world: risk factor aggregation

The most detailed information on the risk profile is provided when all underlying risk factors can be modelled simultaneously. On the Figure 1.1 below, individual risk factors are grouped into N products (price curves), $P_1, \ldots, P_N$, and M obligors (equities) $O_1, \ldots, O_M$. An organisation's exposure to changes (returns) of products' prices is the source of market risk, and its exposure to changes of obligor credit quality is the source of credit risk.

Risk factor aggregation methodology scheme

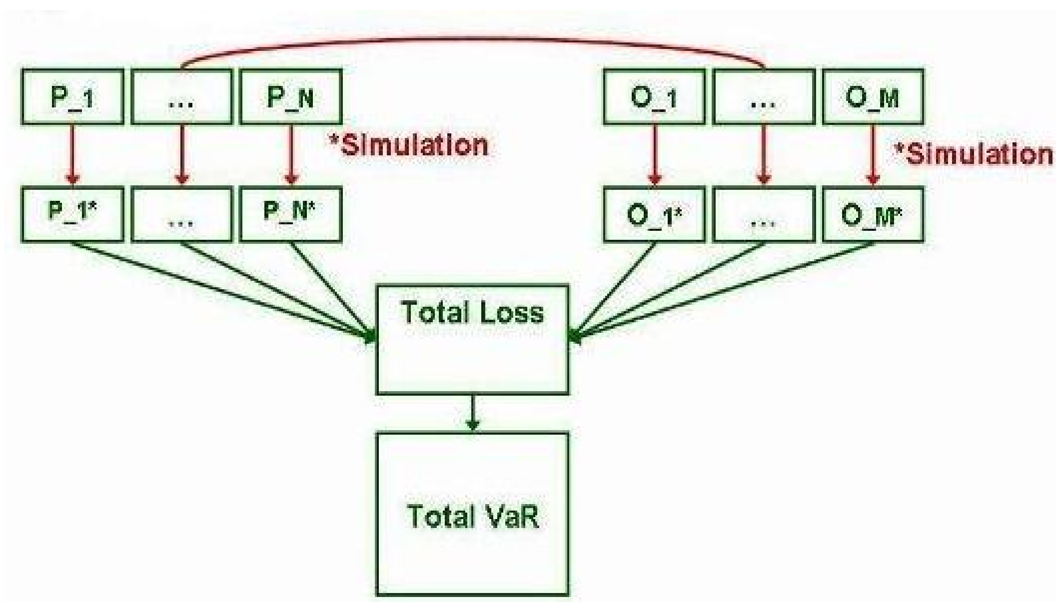

FIGURE 1.1: Risk factor aggregation methodology scheme. *Individual risk factors are grouped into N products (price curves), $P_1, \ldots, P_N$, and M obligors (equities) $O_1, \ldots, O_M$. All risk factors' values are simulated jointly, i.e. according to a common multivariate dependence structure - marked by the red curve. The simulated values are marked with a "star" index: $P_1^*, \ldots, P_N^*$ and $O_1^*, \ldots, O_M^*$.*

The methodology developed in this report was required to be compatible with the CreditMetrics methodology of measuring credit risk [Gup97]. The CreditMetrics methodology is based on Merton's approach to the firm value, which provides a link between the prices of equity and debt instruments issued by a particular obligor. By definition, an obligor's default occurs when the firm value falls below some pre-defined level, referred to as "default threshold" [1]. This definition originally applies to an issued debt, but the link between an obligor's equity and its credit condition is assumed to hold for other forms of debt. Therefore, the changes (returns) of obligors' equity serve as proxy for changes of their credit quality, and obligors' equity returns become the credit risk factors.

In this ideal world, all risk factors' distributions are known, as well as the multivariate dependence structure among them. Therefore, potential values of all risk factors can be simulated jointly (Monte-Carlo). The simulated values can be further processed - added, multiplied, translated to default states, etc. Processing the simulated values can provide information on potential gains/losses resulting from the exposure to a particular risk factor, to the entire portfolio with arbitrary portfolio weights, or to derivative instruments whose pay-offs are functions of the simulated factors' values. The simulated values can be also aggregated, to provide the information on potential gains/losses of the whole organisation exposed to both market risk and credit risk (total loss). A sample of potential values of the total loss form its distribution, to which risk measures - like Value at Risk - can be applied (total VaR).

However, this ideal world has one serious disadvantage: it becomes too complex for a large number of factors. The reason is that for modelling a large number of factors simultaneously, the multivariate dependence concept becomes very complex, and the computational load increases enormously. Even the most standard multivariate dependence concepts (e.g. multivariate normality), require the variance-covariance matrix as input parameters (or at least the lower-triangular matrix of its Cholesky decomposition). Nonetheless, the number of parameters increases geometrically with the increase of the number of risk factors. A separate problem is to ensure that the input variance-covariance matrix is positively-definite.

Given the high-dimensionality problem, risk factor aggregation is only accessible for a small-size model, and is used for the comparison of the methodological approaches (i.e. as a benchmark methodology). The small-size model does not entirely reflect the business environment of a large-size organisation, but it helps identify the sources of possible pitfalls for other methodologies.

1.1.2 The simple world: risk measure (VaR) aggregation

For a large-size organisation, whose books comprise a large number of portfolios, the demand for a simpler aggregation methodology arises naturally. For this purpose, a way of aggregating risk measures (e.g. VaRs) can be considered.

[1] By definition, this level is such that the obligor cannot meet the par payment at maturity or coupon payments of its issued debt. However, given that other forms of debt can be more common for an organisation's obligors, the link between an obligor's equity and its credit condition is assumed to hold whatever the form of an obligor's debt - which is an obvious simplification.

An example of a formula aggregating the risk measures can be found in [Jor], and is also recalled in [OK07]. The formula results from the covariance (correlation) matrix approach to portfolio analysis, providing a closed-form solution for portfolio VaR:

$$VaR_p = \sqrt{\sum_i \sum_j \rho_{ij} VaR_i VaR_j},$$

where VaR_p is the portfolio VaR, $i, j = 1, \ldots, N$, N is the number of individual risk factors (basic portfolio level), books of portfolios or entire risk segments (higher portfolio hierarchy levels), and ρ_{ij} is the correlation coefficient between risk factors i and j, or between the profits/losses of portfolios i and j by given exposures. Here, instead of aggregating risk factors, the aggregation concerns the risk measures - first of individual risk factors, and then of portfolios consisting of them, and books of portfolios - while advancing in the portfolio hierarchy. The final stage of risk measure aggregation can be the aggregation of market VaR and credit VaR.

VaR aggregation methodology scheme

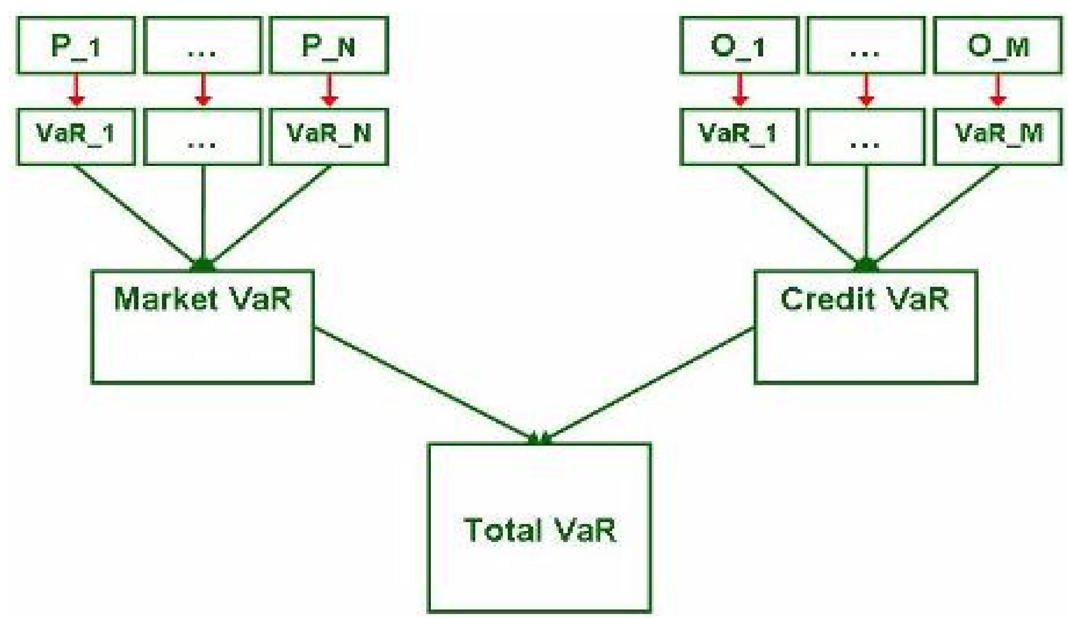

FIGURE 1.2: VaR aggregation methodology scheme. *Individual risk factors are grouped into N products (price curves), $P_1, \ldots, P_N$, and M obligors (equities) $O_1, \ldots, O_M$.*

The VaR aggregation methodology scheme according to the formula explained above is shown on Figure 1.2. As in the previous methodology, all individual risk factors are grouped into N products (price curves), $P_1, \ldots, P_N$, and M obligors (equities) $O_1, \ldots, O_M$. Also -

again - the organisation's exposure to changes (returns) of products' prices is the source of market risk, and its exposure to changes (returns) of obligors' equities is the source of credit risk. VaRs are calculated for each individual risk factor, and then aggregation of VaRs continues while advancing in the portfolio hierarchy until market VaR and credit VaR are aggregated.

However, the use of formulae as above, can be subject to some assumptions regarding the properties of the underlying factors' distributions, or the way of computing the relevant risk measure. For example, the formula above is based on the assumption that all underlying random variables (risk factors) are normally distributed[2], and the Value at Risk is computed using the delta-normal method, in which an individual risk factor's VaR is expressed as a scalar of its standard deviation (volatility).

The high-dimensionality problem and the need for reducing the computational load can be also an argument to use a simpler computational method for the risk measure itself, rather than the simulation-based method. For example, using the delta-normal method instead of simulation reduces the computational load significantly (bearing in mind that it requires strong assumptions on the underlying risk factors' distributions). For more details on methods of computing Value at Risk and their implementation see [Jor].

1.1.3 The compromise: loss aggregation

The methodology proposed in this report is loss aggregation, being a compromise between high dimensionality on one hand, and demand for accuracy on the other. The number of risk factors is just too large for the risk factor aggregation to be handled. As for VaR aggregation, the point at which its accuracy might not be sufficient is the distribution shape and the dependence structure among random variables.

As mentioned before, VaR aggregation as shown above assumes the normal distribution of all underlying risk factors. This, however, is hardly ever the case, and risk factors usually display fatter tails. What is more, VaR aggregation assumes that the dependence between the random variables is fully reflected by the correlation coefficients. This may not be the case either. The historical data samples used in this report show that, especially in the context of aggregating market risk and credit risk, correlation coefficients omit important information on coincidence of extreme loss values.

These two facts forced the development of a "compromise" methodology, which would be applicable to a large number of factors, and that also would ensure some more accuracy - at least as far as it concerns the dependence structure between market risk and credit

[2] In general, the assumption is that the distribution of the underlying random variables is stable. The stability property means that, given the distribution type of the underlying random variables, their weighted sum has the same distribution type - however, different parameters are allowed. The normal distribution is a common example of a stable distribution (distributions having this property are sometimes referred to as sum-stable, for reference see [Nol07]).

risk. For these two risk types, the historical data sample used herein shows that their dependence structure is significantly different from normal (Gaussian). Therefore, using VaR aggregation could potentially lead to a faulty description of the overall risk profile.
The Figure 1.3 below illustrates the loss aggregation methodology scheme, with the same individual risk factors and the exposures to risk as in the previous methodologies. It is assumed that the organisation does not benefit from credit-related profits, so it is only exposed to possible credit losses - referred to as “total credit loss“. The market activity can generate both. Given that gains are understood as negative losses, the market activity profits/losses will be referred to as “total market loss“.

Loss aggregation methodology scheme

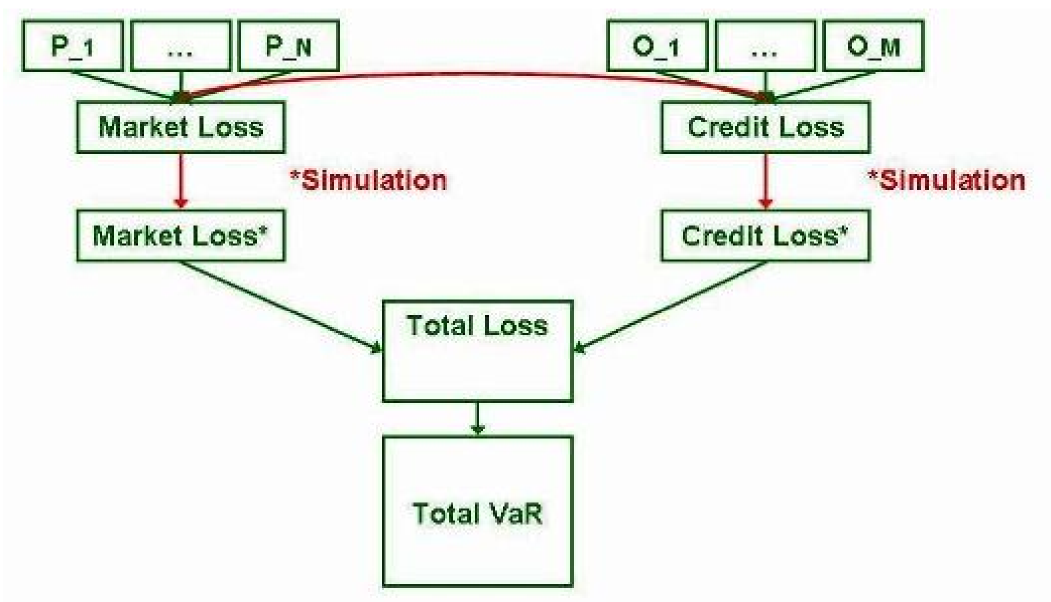

FIGURE 1.3: Loss aggregation methodology scheme. *Individual risk factors are grouped into N products (price curves),* $P_1, \ldots, P_N$*, and M obligors (equities)* $O_1, \ldots, O_M$*. Values of total market loss and total credit loss are simulated jointly, i.e. according to a common bivariate dependence structure - marked by the red curve. The simulated values of total market loss and total credit loss are marked with a “star“ index.*

Loss aggregation aims at producing a bivariate historical sample of total market loss and total credit loss values. This sample is a result of applying a particular combination of exposures to the historical values of individual risk factors. It means that the sample is strictly related to a particular combination of exposures. The bivariate sample provides sufficient statistical data to define the bivariate dependence structure. Simulation of joint values using this very dependence structure and the respective marginal distributions of the two variables, provides information on potential profits/losses of the whole organisation exposed to both market and credit risk (referred to as “total loss“). A sample of values of the total loss form its distribution, to which risk measures (e.g. VaR) can be further applied.

Chapter 2

Loss aggregation: the small-size model

The primary focus of this chapter is to compare the risk aggregation methodologies: risk factor aggregation, loss aggregation, and risk measure (VaR) aggregation using the small-size model (according to the systematics provided in Chapter 1). The emphasis is on loss aggregation as the methodology proposed for implementing in a large-size organisation.
In order to ensure the comparison between loss aggregation and other methodologies of risk aggregation, it is explained on a small-size model with four price curves, four obligors, and equally distributed exposures. The small-size model is built so as to reflect the business environment of a large-size organisation as appropriately as possible.
The chapter is organised as follows: the first section presents the small-size model, on which the methodology details are explained. Section two includes the mathematical framework specific for loss aggregation. Section three focuses on the risk measurement process.

2.1 The model

Despite that the loss aggregation methodology was originally developed to be implemented on a larger scale, the high-dimensionality problem forces that many of its characteristics are analysed on a smaller scale. These characteristics comprise not only the references to other risk aggregation methodologies, but also issues related to dependence modelling, stress-testing and verification of the model assumptions. In this way, potential insufficiencies of information on the risk profile can be captured.
The hypothetical small-size model mirrors the business environment of an organisation trading 4 products (spot only): 2 of type 'A' (P1 and P2),and 2 of type 'B' (P3 and P4). The organisation is exposed to credit risk of 4 obligors: 2 of sector 'C' (O1 and O2), and 2 of sector 'D' (O3 and O4). For model simplicity, as well as to conform to the large-size

business environment, no gains resulting from changes in credit quality are allowed, and the exposures related to obligors are assumed to have the form of accounts receivable. The small-size business model is build so as to reflect the correlation scheme of a larger-size one, as shown on the Figure 2.1 below.

Correlation scheme for hypothetical small-size business model

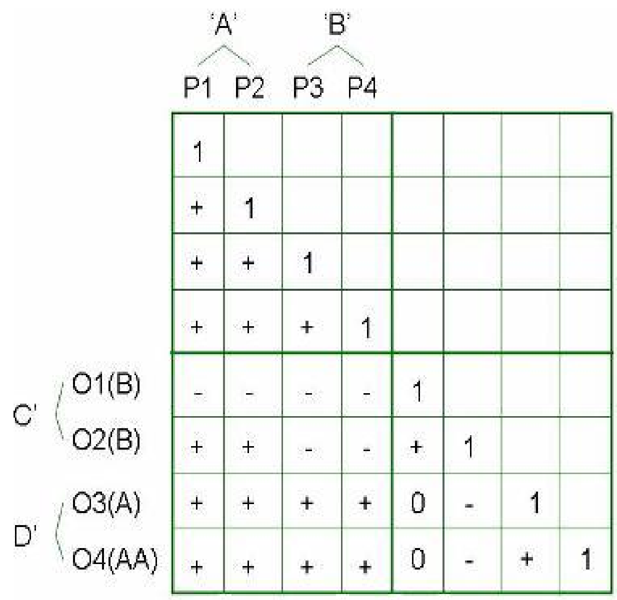

FIGURE 2.1: Small-size hypothetical business model. *Pluses and minuses indicate whether the correlation between spot price curves' returns (products) and spot equity returns (obligors) is positive (+), negative (-), zero (0) or (1); Obligors' ratings given in brackets;*

The total of the exposures related to products and the total of the exposures related to obligors are equal and total to USD 1M each. The exposures are equally split among products (USD 0.25M for each of the products) and among obligors (USD 0.25M for each of the obligors). The question arises whether the total of the exposures related to products and the total of the exposures related to obligors should be the same. In general the answer is no, since the activity on the market is not directly financed with accounts receivable from obligors. However, for the purpose of this analysis they are the same, so that the scale of market and credit activity is comparable, and so are the risk figures.

2.1.1 Building the historical data sample

The first step to loss aggregation is creating a bivariate historical data sample of total market loss and total credit loss. The history of total market loss and total credit loss for the small-size model was built based on existing historical data and the CreditMetrics methodology for measuring credit risk[1].

The historical data sample for the small-size model are time series of approximately 2 years (522 observations) of annual frequency data, for each product and each obligor. This data sample contains observations related to all risk factors individually (4 price curves' returns and 4 obligors' equity returns, meaning 8 risk factors altogether). For price curves' (logarithmic) returns, the data sample takes the form of table 2.2:

Price curves' annual returns for hypothetical small-size model

Observation/Curve	R_{P1}	R_{P2}	R_{P3}	R_{P4}
1	0.5365	0.4713	0.4747	0.0793
2	0.4264	0.3818	0.3316	0.1922
3	0.2483	0.1577	0.2298	0.3813
...	...	...	...	...
522	0.0796	0.0390	0.4058	0.3024

TABLE 2.2: Price curves' annual returns for hypothetical small-size model. *The first column indicates the observation's number, further columns include annual (logarithmic) returns related to particular price curves on this date. Price curves are identified by products' indexes (P1,P2,P3,P4).*

For obligors' equity returns the data sample initially takes the form of table 2.3A. Later, equity returns are translated into default states, according to the firm value model used in the CreditMetrics methodology.

Obligors' annual equity returns for hypothetical small-size model

Observation/Curve	R_{O1}	R_{O2}	R_{O3}	R_{O4}
1	-0.2937	-1.3603	0.2467	0.1604
2	-0.4199	-0.9403	-0.0280	-0.0020
3	-0.2887	-1.8513	0.2909	0.2173
...	...	...	...	...
522	-0.1988	-1.3603	0.5760	0.2901

TABLE 2.3A: Obligors' annual equity returns for hypothetical small-size model. *The first column indicates the observation's number, further columns include annual (logarithmic) returns related to particular obligor's equity on this date. Obligors' equities are identified by obligor indexes (O1,O2,O3,O4).*

[1]Data made available by BP Oil International. Consistency with CreditMetrics methodology was the model requirement.

Default indication for obligors, for hypothetical small-size model

Observation/Curve	D_{O1}	D_{O2}	D_{O3}	D_{O4}
1	1	1	0	0
2	1	1	0	0
3	1	1	0	0
...	...	...	...	...
522	0	1	0	0
Threshold	-2.7969	-0.7417	-0.7614	-0.5467

TABLE 2.3B: Default indication for obligors, for hypothetical small-size model. *The first column indicates the observation's number. Other columns indicate if there was default recorded for a particular obligor on this date. Default states are indicated with 1, non-default states with 0. For clarity, the last row contains default thresholds. Obligors' equities are identified by obligor indexes (O1,O2,O3,O4).*

However, a simplified 2-state model (default and non-default) is considered here - as opposite to the original multi-state model, in which obligors are allowed to migrate across states defined by rating. Here, they are only allowed to move to the default state, or stay within the same rating group. A default is recorded if an obligor's equity (logarithmic) return falls below some pre-defined threshold, and the recovery rate is assumed to be zero (the entire amount is lost by default). The equity returns translated into default states take the form of table 2.3B.

This way of assigning default or non-default states to historical returns is obviously not ideal, the most doubtful being the question "Was there really a default then?". Indication of default states by obligors' equity returns is a subject having a long record of analysis in the history of economics, and it is far beyond the scope of this research to discuss it. The default concept used here was selected because practically no defaults were recorded within the period covered by the data sample, and the mere fact that - again - the same concept is used to predict potential defaults in the future.

Whereas the occurrence of real defaults is questionable, the way of accounting for losses if the equity returns are below the default threshold can be understood as a kind of barrier pay-off function. Since the real defaults and their consequences (recovered payments) are difficult to monitor, such a barrier pay-off function - rather similar to credit derivatives' pay-off functions - serves as a simplified credit loss model. Thus, the credit loss model applied to the past data conforms to the credit loss model applied to the simulated data, according to the CreditMetrics methodology.

2.1.2 Processing data

It is important to explain the differences in how the above data are used, with respect to each of the methodologies presented at the beginning of the chapter. For the time being, the historical data sample provides information on all individual risk factors (price curves' returns and obligors' equity returns).
For risk factor aggregation, processing data takes the following steps:

1. Establishing multivariate dependence structure among all individual risk factors;

2. Simulating risk factors' values jointly;

3. For credit risk factors, translating simulated values into default states;

4. Computing projected values of total loss by taking the sum of simulated loss values for individual risk factors, multiplied by their respective current exposures;

5. Calculating (joint) risk figures for projected values of total loss.

Risk factor aggregation first involves the data as of tables 2.2 and 2.3A. For the small-size model, the multivariate dependence structure for all individual risk factors proved to be the t-copula with the degrees of freedom parameter ν equal to 14.49. This dependence structure is not exactly the same as multivariate normality, which corresponds to the infinite degrees of freedom parameter case. The degrees of freedom parameter correspond to the strength of tail dependence - the lower its value, the stronger the tail dependence. It must be stressed that the degrees of freedom parameter is much lower among market risk factors ($\nu_M = 6.03$, meaning that the tail dependence is significantly stronger among them), than among credit risk factors ($\nu_C = 16.52$). Also the dependence structure defined by the t-copula fits much better to market risk factors (logarithmic likelihood equal to 1114.5) than to credit risk factors (logarithmic likelihood equal to 264.46). The goodness-of-fit to market risk factors dominates the dependence structure, resulting in the overall logarithmic likelihood equal to 1397.57, but compromises on the degrees of freedom parameter value.
Simulation from multivariate t-copula is not complicated (for reference, see [McN05]), and produces a joint sample of potential values of all risk factors. These, in turn, can be processed to provide information on potential losses of the entire organisation, resulting from its exposure to both market risk and credit risk. For this particular purpose, simulated values of obligors' equity returns need to be translated into potential default states - as in table 2.3B. However, table 2.3B concerns translating historical equity returns into default states, not the simulated ones. This is a vital difference between risk factor aggregation and loss aggregation. Finally, simulated risk factor values (price curves' returns and default indicators) are multiplied by their respective exposures to generate projected values of total

loss. Risk measurement is applied to the distribution formed by projected values of the total loss.
Loss aggregation means that market risk and credit risk are aggregated into total market loss and total credit loss at the point of building historical data sample. Opposite to risk factor aggregation, loss aggregation involves data as of table 2.2 and table 2.3B, because translation to default states is done to historical obligors' equity returns (not the simulated ones). Processing data for loss aggregation involve the following steps:

1. Creating a bivariate historical data sample of total market loss and total credit loss, by multiplying historical risk factor values (price curves' returns and default indicators) by their respective current exposures, and taking their respective sums;
2. Establishing bivariate dependence structure between total market loss and total credit loss;
3. Simulating loss values jointly;
4. Taking the sum of simulated values will result in a projection of total loss incurred by the organisation;
5. Calculating (joint) risk figures for projected values of total loss;

Referring to data in table 2.2, the total market loss is computed as:

$$L^M = \mathbf{R} \cdot (\mathbf{E}^M)^T$$

where L^M is the total market loss, $\mathbf{R}$ is the vector of returns, and $\mathbf{E}^M$ is the vector of exposures to market risk factors. Since the total of the exposures related to products is USD 1M, and the exposures are equally split among products (USD 0.25M for each), then:

$$\mathbf{E}^M = (250.00; 250.00; 250.00; 250.00);$$

Referring to data in table 2.3B, the total credit loss is computed as:

$$L^C = \mathbf{D} \cdot (\mathbf{E}^C)^T,$$

where L^C is the total credit loss, $\mathbf{D}$ is the vector of default indicators, and $\mathbf{E}^C$ is the vector of exposures to credit risk factors. Since the total of the exposures related to obligors are equal is USD 1M each, the exposures are equally split among obligors (USD 0.25M for each), then:

$$\mathbf{E}^C = (250.00; 250.00; 250.00; 250.00);$$

The pair (L^M,L^C), standing for total market loss and total credit loss, forms a 2-dimensional random variable. Therefore, loss aggregation can be understood as reducing a large number of random variables into a bivariate case. In this operation, the output 2-dimensional random variable consists of two weighted sums of the original random variables grouped into two categories, according to their identification as a market risk factor or as a credit risk factor.
Finally, for VaR aggregation (in general: risk measure aggregation), processing data takes the following steps:

1. Computing total market loss and total credit loss as for loss aggregation;
2. Simulating loss values independently;
3. Computing VaR (in general: risk measure) for each of the risks independently;
4. Aggregating VaRs (risk measures) according to the VaR aggregation formula to obtain joint risk figures.

The main difference between loss aggregation and VaR aggregation is that the latter does not account for the dependence structure between total market loss and total credit loss, and uses only the information provided by the correlation coefficient (at least, if the formula is used as in [Jor]). The second point is that VaR aggregation assumes the underlying variables to be normally distributed, whereas loss aggregation allows basically any distribution type. herein, the difference between loss aggregation and VaR aggregation applies to the very top of the risk structure, i.e. at the point of aggregating market risk and credit risk, and providing joint risk figures for the two. Assuming that the risk figures computed independently for each risk type are not biased with errors resulting from other techniques, the comparison focuses strictly on the importance of the dependence structure.
Further analysis focuses on loss aggregation and - in particular - fitting a bivariate copula-based dependence structure between total market loss and total credit loss. Special attention is drawn to its accuracy and to the assumptions of the underlying random variables' distribution types and tail shapes. References to risk factors aggregation and VaR aggregation are made whenever a comparison is useful.

2.2 Loss aggregation - general mathematical framework

The mathematical concept behind the aggregation of market and credit risk is based on the copula theory. For the bivariate case, the dependence structure defined by a copula is

potentially the most accurate - as it is not an “average“ type of dependence among many different dependence structures for each pair of factors.
The copula theory allows the joint distribution of total market loss and total credit loss to be is decomposed into two parts: marginal distributions of both random variables, and dependence structure (pattern) given by a copula. For details of decomposition of joint distributions see Appendix A.

2.2.1 Marginal Distributions

Marginal distributions are the univariate distributions of each individual random variable. One of the two approaches can be adopted here: either the empirical distribution is taken, or an assumed distribution type. On one hand, the copula parameters are estimated non-parametrically, which means that they are estimated based on empirical distributions of total market loss and of total credit loss (see Appendix B for details). On the other hand, considering a fixed distribution type to represent the marginal distribution of total market loss, can be an attractive reference point to a methodology, in which such a distribution type is a part of the model assumptions (e.g. VaR aggregation methodology as presented in the previous section, assuming the sum-stable distribution type of underlying individual risk factors). For example, for the normal distribution, a linear combination of logarithmic returns is - by its properties - also normally distributed with appropriately adjusted parameters. This reasoning is key when applying the delta-normal method of VaR calculation.

Example of market risk factor's ($P1$) logarithmic return density

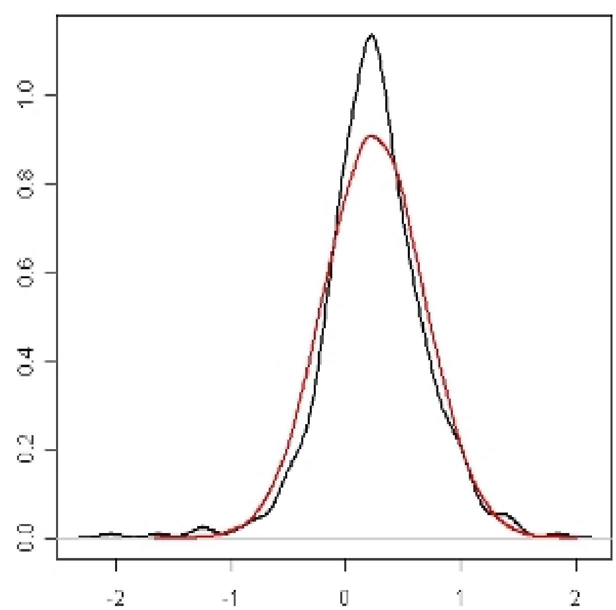

FIGURE 2.4: Example of market risk factor's ($P1$) logarithmic return density. *Black line - empirical density of historical sample of logarithmic returns for* $P1$*. Red line - normal distribution density with the same mean and standard deviation as the historical sample for* $P1$*.*

However, in the small-size model, the individual market risk factors' distributions (price curves' returns) are not normal. Statistics for Jarque-Bera and Shapiro-Wilk tests force to reject the null hypothesis that the distribution type is normal, with the p-value below 0.05 (for the construction of the test see [JBC94] and [Roy82], respectively). What is more important is that the difference in distribution shape involves the shape of tails - the individual market risk factors' distributions tails show multi-modalities in the right-tail area. This effect is dangerous if extreme losses are to be captured.

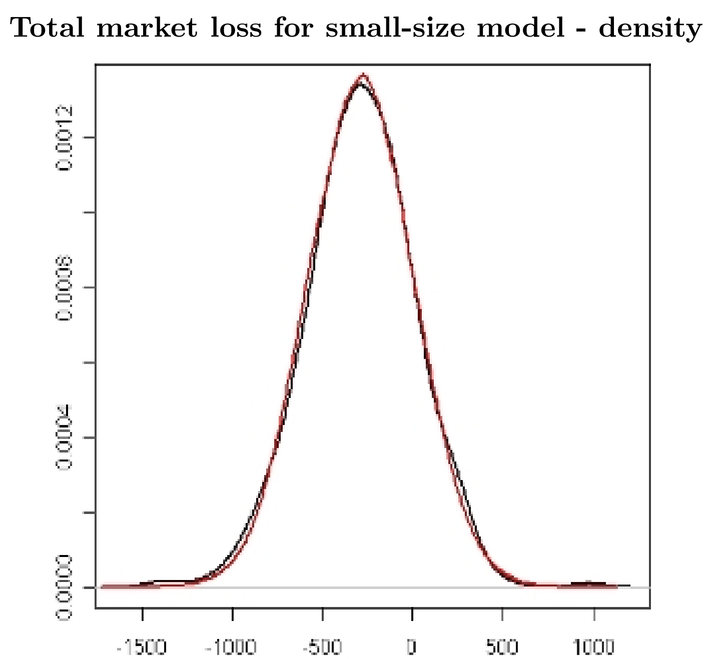

FIGURE 2.5: Total market loss for small-size model - density. *Black line - empirical density of historical sample of total market loss. Red line - normal distribution density with the same mean and standard deviation as the historical sample for total market loss.*

As for the shape of the total market loss's distribution (Figure 2.5), it appears to be well approximated by the normal distribution. Unfortunately, the most significant difference in the distribution shapes is again in the right tail area (extremely high losses). Total market loss density exhibits a tail slightly fatter than normal. This effect (visible on Figure 2.6) worsens the normality tests statistics.

What is more, it cannot be guaranteed that the distribution shape of total market loss can be approximated by the normal distribution for any other combination of exposures, or a different selection of risk factors for the small-size model. In the general case, replacing the total market loss distribution with the normal distribution can lead to a distorted information on the risk profile.

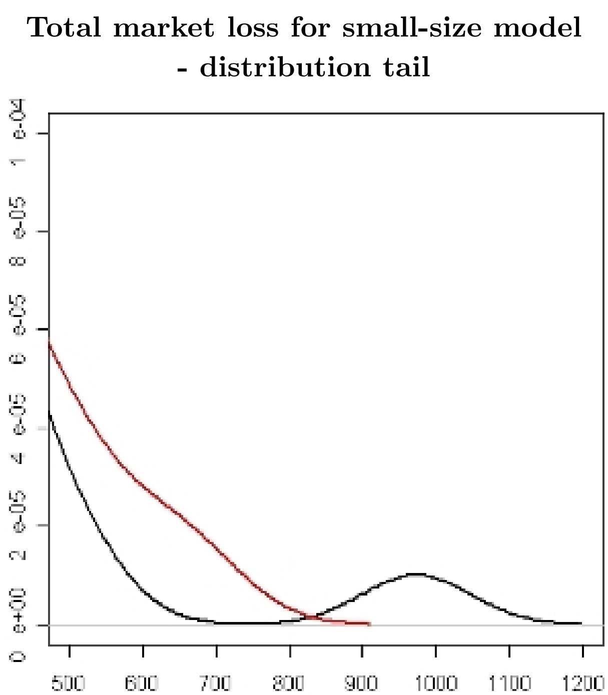

FIGURE 2.6: Total market loss for small-size model - - distribution tail. *Black line - empirical density of historical sample of total market loss. Red line - normal distribution density with the same mean and standard deviation as the historical sample for total market loss.*

Individual credit risk factors (obligors' equity returns) are not normally distributed either (see Figure 2.7). The normality test statistics speak firmly against the null hypothesis of the distribution normality, mostly due to the tail shape and multi-modalities.
As for the total credit loss, its marginal distribution shows visible discontinuities (see Figure 2.8). This very characteristic is a direct consequence of the model's small-size. For a small number of obligors (and arbitrary exposures), the range of possible total credit loss values is very narrow (0,1,2,3 or 4 obligors defaulting). If the same exposures are assigned to each of the obligors, the set of possible values is even more limited (0, 1, . . . , 4 times the exposure).

Example of credit risk factor's ($O1$) logarithmic return density

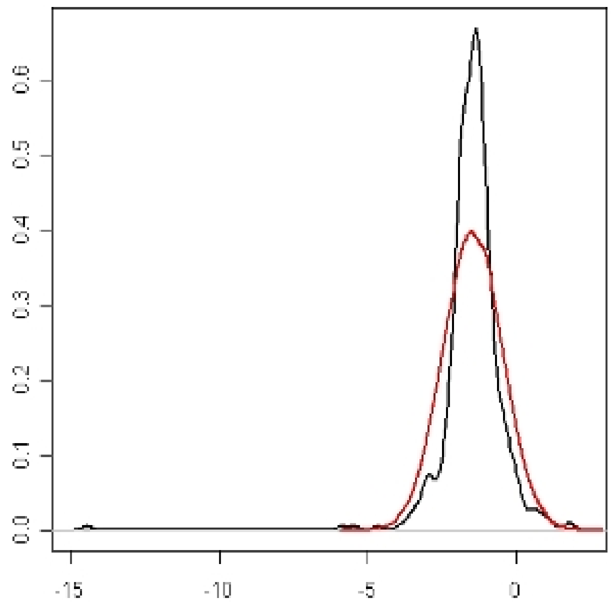

FIGURE 2.7: Example of credit risk factor's ($O1$) logarithmic return density. *Black line - empirical density of historical sample of logarithmic returns for $O1$. Red line - normal distribution density with the same mean and standard deviation as the historical sample for $O1$.*

Total credit loss for small-size model - approximate density

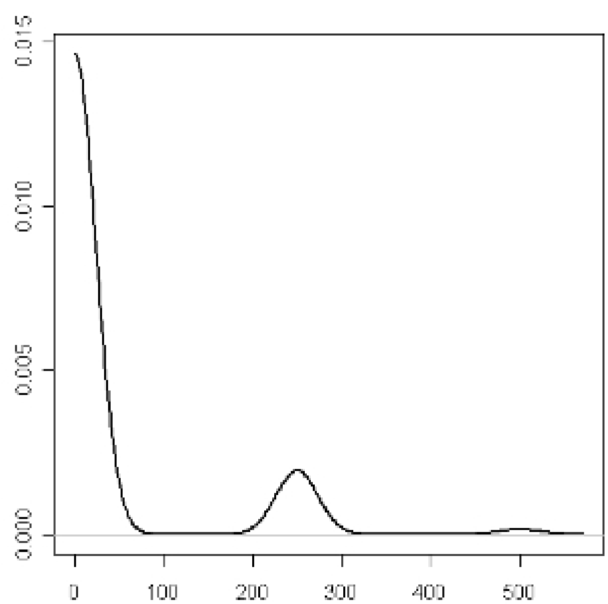

FIGURE 2.8: Total credit loss for small-size model - approximate density. *Black line - empirical density of historical sample of total credit loss.*

The fact that one of the two random variables involved (total credit loss) has a discontinuous marginal distribution has important consequences for the dependence structure. From the mathematical point of view, the dependence structure defined by a copula originally concerns random variables, of which the marginal distributions are continuous. Otherwise, either the marginal distribution needs to be approximated by a continuous one, or the copula properties need to be generalised (so that they account for arbitrary marginals) - or a combination of the two. For the small size model, a combination of the two is approached. Moreover, non-standard marginal's characteristics are a strong argument for leaving the marginal distributions as empirical, so that the reasoning behind the loss aggregation methodology is mathematically justified.
Besides the marginal distributions' shapes discussion, it is also important to ensure that the random variables are cleared from serial dependence. Serial dependence statistics for individual risk factors (Ljung-Box test, see [Box78]) do not allow us to reject the null hypothesis of independence up to order 5 (a natural cycle of 5 working days). The statistics' p-value is around 0.6, which means that at a decent confidence level (say the confidence level of 90%), the test does not allow us to reject the null hypothesis, stating that there is no serial dependence. By the test construction, the test result doesn't ensure that there is no serial dependence at all, but - given such statistics - the time series for individual risk factors are assumed to be serially independent. The null hypothesis of independence is stronger for market risk factors than for credit risk factors, and obviously stronger for higher orders than for lower orders. Similar statistics are obtained for total market loss and total credit loss, but for these two loss aggregation variables, the independence hypothesis has a weaker ground (the p-value around 0.4).
For the reasons mentioned above: the shape of the distribution failing to be normal (total market loss), or exhibiting discontinuities (total-credit loss), the marginal distributions for total market loss and total credit loss are left as empirical, and are assumed to be serially independent.

2.2.2 Copulas and dependence structure

Copulas are a very useful tool for decomposition of joint distributions. However, the properties of the marginal distributions (particularly the total credit loss) excluded using standard copula types, and forced to search for alternative dependence structure. This is what the current research aims at - providing a reliable dependence structure between total market loss and total credit loss for the small-size model. The ultimate concept should allow for the loss aggregation methodology to become mathematically accurate for the small-size model, and its characteristics to be analysed using a reliable benchmark.

For the time being, a simulation technique has been developed, whose primary goal is to reproduce the dependence pattern visible on scatter-plot 2.9 below. The properties of this technique are being analysed at the moment, in view of defining a proper bivariate distribution and - later - a copula.

Historical values of total market loss and total credit loss for small-size model - scatter-plot

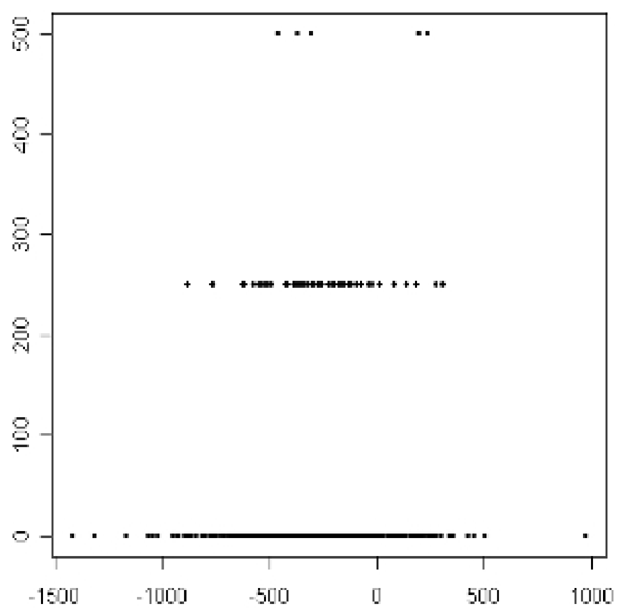

FIGURE 2.9: Historical values of total market loss (X-axis) and total credit loss (Y-axis) for small-size model, in USD - scatter-plot.

The scatter-plot (Figure 2.9) illustrates the dependence pattern between the total market loss and the total credit loss for the small-size model. The total credit loss shows visible cumulation at certain levels. These levels are directly related to the number of obligors defaulting, and the exposures related to each obligor. The plot above indicates maximum two obligors defaulting, and if an exposure of USD 0.25M is related to each of them, the resulting loss is USD 0.5M.
The total market loss is rather evenly distributed at each of the levels, with the mean increasing slightly from level to level. This means that higher credit loss values coincide with higher (on average) market loss values.
For fitting a copula, the values of total market loss and the total credit loss need to be transformed into values of their respective distribution functions by the probability integral

transform (see Appendix B for details). For the transformed values, the dependence pattern looks very similar (only weighted by probability mass relevant for total credit loss's cumulation levels).

Historical values of respective cumulative distribution functions of total market loss and total credit loss for small-size model - scatter-plot

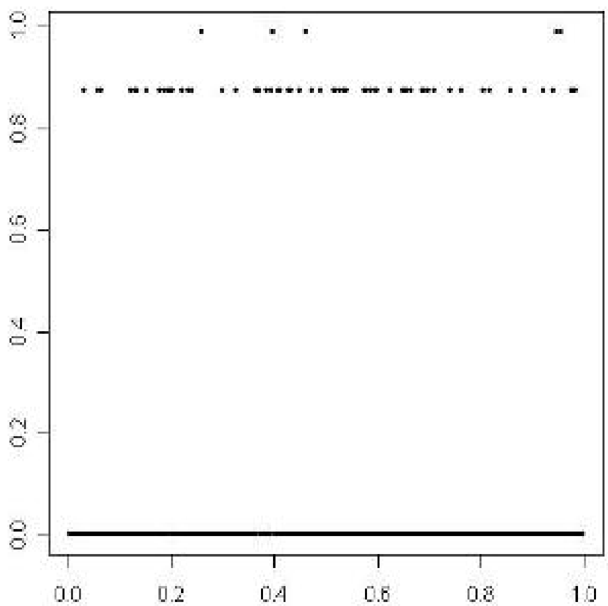

FIGURE 2.10: Historical values of respective cumulative distribution functions of total market loss (X-axis) and total credit loss (Y-axis) for small-size model - scatter-plot. *Values of total market loss and total credit loss transformed into values of their respective cumulative distribution functions by probability integral transform.*

2.3 Risk measurement

For the loss aggregation methodology, the risk measurement process is based on joint simulation of total market loss and total credit loss values.

First, 100,000 pairs (total market loss and total credit loss) are simulated. These pairs are simulated values of the two random variables, having the dependence structure given by the hypothetical copula, and the marginal distributions given by their respective empirical distributions (see Appendix C for simulation details). If the technique mentioned in the previous section proves to be a proper copula, the simulation reproduces the dependence pattern between total market loss and total credit loss very accurately (as on Figure 2.11).

Then, for each pair, the loss values are added. In this way, projected values of total loss - related to the current market activity and the current obligor portfolio - are obtained. Projected values of total loss form the total loss distribution, which is the start point of calculating joint Value at Risk and Expected Shortfall.

Historical and simulated values of total market loss and total credit loss for small-size model - scatter-plot

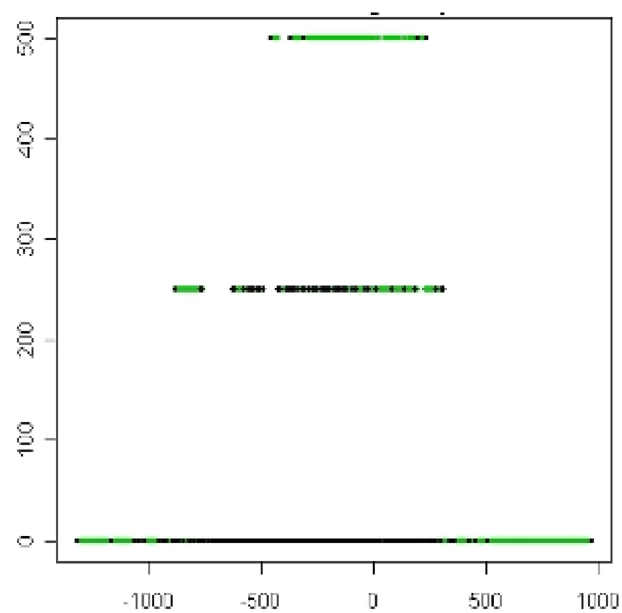

FIGURE 2.11: Historical and simulated values of total market loss (X-axis) and total credit loss (Y-axis) for small-size model, in USD - scatter-plot. *Historical losses marked with black points, simulated losses - with green points, simulation using the hypothetical copula.*

The three methodologies presented in the previous chapter, are compared in terms of their impact on risk measurement. The primary risk measure used is Value at Risk (VaR), supported by the Expected Shortfall (ES).

The widespread use of VaR in the financial industry is partly attributable to its conceptual simplicity (see Appendix D for formal definitions). Additionally, in the financial industry, regulators use VaR as a starting point to calculate economic capital. The economic capital calculated by using the VaR at the $100\alpha\%$ confidence level, corresponds to the capital needed to keep the firm's default probability below $100(1-\alpha)\%$. That explains why risk practitioners use VaR as a measure to influence the firm's risk profile.

In this report, the confidence level is set as 99.9%, and Value at Risk is calculated as the 0.999-th quantile of the total loss distribution (bearing in mind that gains are treated as negative losses). Out of $100,000$ values of total loss ordered increasingly (which form the approximated total losses distribution), the 0.999-th quantile is the $99,900$-th point.

Importantly, VaR provides no information about the magnitude of losses which might exceed the specified threshold $100\alpha\%$. If the shape of the loss distribution exhibits some non-standard features (fat tails, multi-modalities, jumps), the actual loss can be substantially higher then the VaR estimate.

Expected shortfall can be defined as the average (i.e. the expected value) of losses that exceed the VaR confidence level. The metric has an appealing informative power about the magnitude of losses exceeding the confidence level. This is why the ES metric values are also provided, to support the numerical results, despite a very high confidence level (with the confidence level set at such a high percentile, both metrics return similar values except for some very special cases).

The VaR and ES figures are presented in Table 2.12 below. The first column contains market VaR (ES) - for total market loss, credit VaR (ES) - for total credit loss, and the joint VaR (ES) - for total loss (i.e. the sum of total market loss and total credit loss) - when all risk factors were simulated jointly, by means of risk factor aggregation (benchmark methodology). The second column contains the same risk measures for loss aggregation, i.e. when the aggregate variables (total market loss and total credit loss) were simulated jointly, with the hypothetical copula, and their respective marginals. The third column contains market VaR (ES) and credit VaR (ES) for total market loss and total credit loss simulated independently, and the joint risk figures (VaR and ES) were calculated by means of VaR aggregation formula, with the correlation coefficient between total market loss and total credit loss equal to -0.0234 (Kendall's τ equal to -0.0031).

VaR and ES figures for small-size model

VaR (ES)	Risk factor aggregation	Loss aggregation	VaR aggregation
Market VaR (ES)	0.7167 (0.8309)	0.6958 (0.8310)	0.7137 (0.8374)
Credit VaR (ES)	0.5000 (0.5000)	0.5000 (0.5000)	0.5000 (0.5000)
Joint VaR (ES)	0.7988 (0.9225)	0.8059 (0.9117)	0.8618 (0.9652)

TABLE 2.12: Market VaR (ES), credit VaR (ES), and aggregate VaR (ES) for small-size model, in USD M. *Figures calculated by means of the risk factor methodology (column 1), the loss aggregation methodology (column 2), and the VaR aggregation methodology (column 3). Confidence level for VaR* 99.9%.

Provided the technique mentioned in the previous section proves to be a proper copula, the risk figures it generates look reasonable - as compared to the benchmark methodologies. The results in Table 2.12 indicate, that market VaR (ES) values and credit VaR (ES) values are close (market) or even the same (credit) for all three methodologies, the lowest VaR being the one for loss aggregation. The differences occur for joint VaR and ES - risk factor aggregation and loss aggregation return similar values of joint VaR and ES (a promising outcome for the hypothetical copula), whereas VaR aggregation returns visibly higher risk figures.

The VaR and ES figures are the highest for VaR aggregation, despite a negative low correlation coefficient between total market loss and total credit loss. However, looking at Figure 2.9 - the tendency is visible for higher values of total credit loss to coincide with higher average values of total market loss. This indicates that correlation coefficient is not an ultimate dependence measure in this case (neither is Kendall's τ). Within the confidence level, the most adverse combination of total market loss and total credit loss is when the total market loss is slightly above its average, and the total credit loss takes its maximum value (coordinates of the point located the most top-right on Figure 2.9).

For loss aggregation, the joint VaR is approximately 30% lower than the sum of market VaR and credit VaR computed independently. As it was mentioned before, VaR is a starting point for calculating an organisation's economic capital. If risks are allowed to be treated jointly, potential savings in economic capital could add up to a similar percentage, vs. the sum of the economic capital calculated for market risk and credit risk separately.

Finally, to make the analysis complete, it is useful to look at the marginal distributions of total market loss, total credit loss and total loss - both empirical (historical), and simulated - using risk factor aggregation and loss aggregation. The empirical distributions are basically the same as the distributions used for VaR aggregation (in which total market loss and total credit loss are simulated independently).

Figure 2.13 illustrates the total market loss density. The value range for the empirical (historical) and the simulated distribution of total market loss (whatever the methodology), is approximately the same. What is more, the tail range is similar - which translates to similar market VaR and ES figures (in Table 2.12). However, as a consequence of using the hypothetical copula, the shape of total market loss distribution resulting from loss aggregation is visibly multi-modal, and the local maxima reflect the cumulation levels of the total credit loss.

Similarly to total market loss, also the value range for the empirical (historical) and the simulated (whatever the methodology) distribution of total credit loss is approximately the same. This does include the tail range - which again translates to similar credit VaR and ES figures (Table 2.12). For other values, the simulated density curves show more concentrated probability mass around thinner cumulation areas (both for risk factor aggregation and loss aggregation). Nevertheless, all distributions' shapes are similar, with visible cumulation of total credit loss at certain levels.

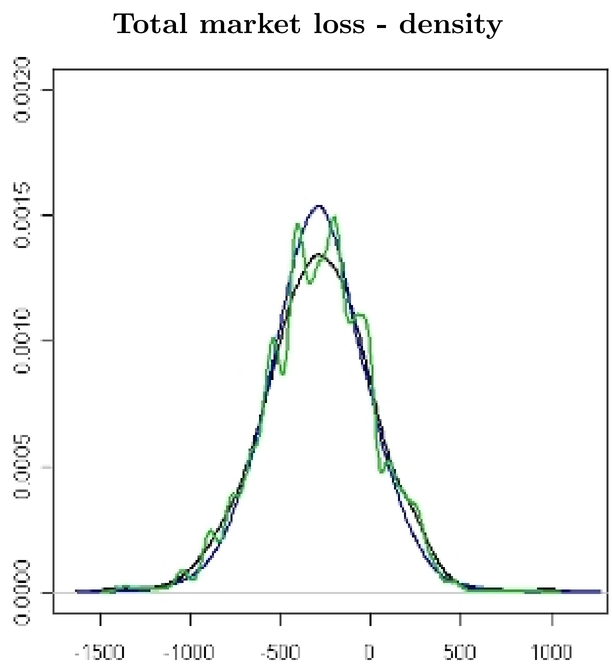

FIGURE 2.13: Total market loss for small-size model, in USD - density. *Black line - historical density, blue line - density simulated using the risk factor aggregation methodology, green line - using the loss aggregation methodology.*

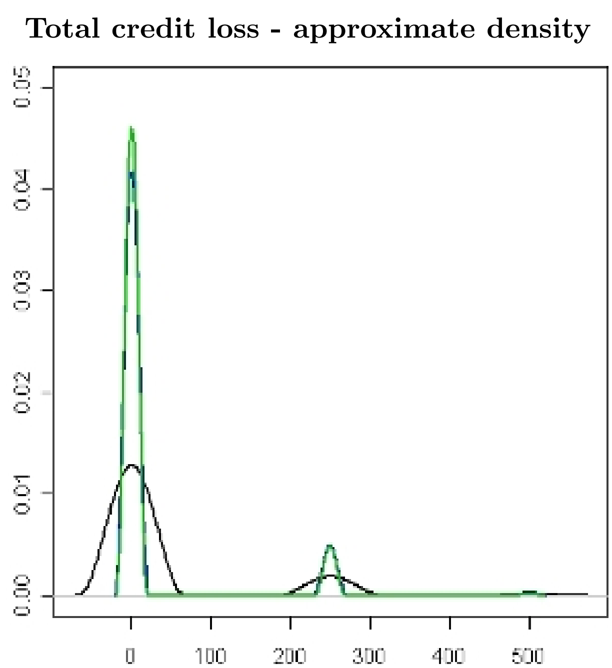

FIGURE 2.14 Total credit loss for small-size model, in USD - approximate density. *Black line - historical density, blue line - density simulated using the risk factor aggregation methodology, green line - using the loss aggregation methodology.*

On the contrary, the value range for the empirical (historical) and the simulated distribution of total loss varies slightly. The historical distribution and the one simulated using loss aggregation have the same tail range, whereas the tail range of the total loss distribution simulated using risk factor aggregation is slightly wider. This translates into joint ES being a little higher for risk factor aggregation - the difference in tail range reaches beyond the confidence level and is therefore not captured by VaR. The total loss distribution resulting from loss aggregation inherits the multi-modal shape from the hypothetical copula used as dependence concept.

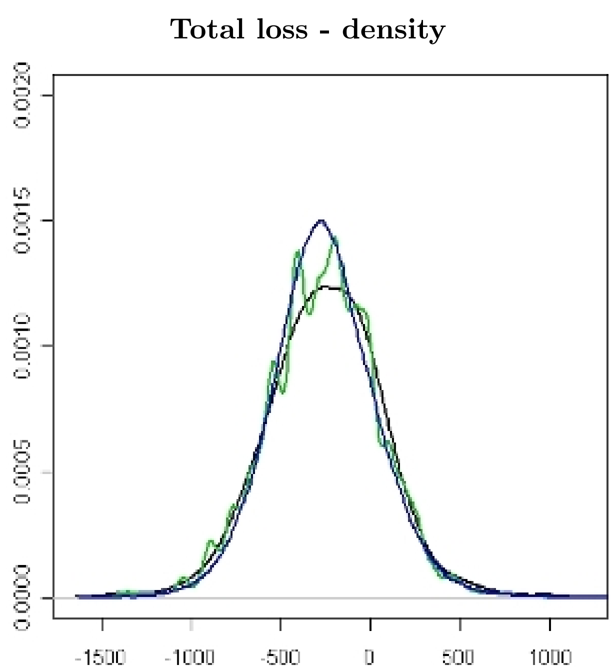

FIGURE 2.15: Total loss for small-size model, in USD - density. *Black line - historical density, blue line - density simulated using the risk factor aggregation methodology, and green line - using the loss aggregation methodology.*

Summarising, both the risk figures and the distributions' shapes indicate, that there seems to be no significant loss of information when loss aggregation with the hypothetical copula as dependence structure is used instead of risk factor aggregation (for the small-size model). However, some differences in VaR and ES figures occur between these two methodologies and

VaR aggregation. For this reason, it is useful to compare behaviour of all the methodologies in more adverse conditions - i.e. by stress-testing. Once the properties of hypothetical copula are fully analysed, the next research step is stress-testing of all the three methodologies.

Chapter 3

Loss aggregation: the large-size model

Loss aggregation methodology was originally developed to be implemented in a large-size organisation. It was also preliminarily tested on a large-size model, encompassing several hundreds of market risk factors (price curves) and credit risk factors (obligors' equities). Although for a model of this size a comparison of all methodologies is hardly possible (risk factor aggregation is not accessible), it is possible to compare the risk measurement process using loss aggregation and VaR aggregation.

Chapter 3 is organised as follows: the first section summarises the characteristics of a large-size model. Section two focuses on the dependence structure between the total market loss and total credit loss. Finally, section three illustrates the risk measurement process, using a numerical example.

3.1 The large-size model

The large-size model was build based on data made available by BP Oil International[1], encompassing a segment of its market activity (several hundreds of price curves), and a group of obligors (several hundreds of obligors' equities).

The organisation's performance in this business segment is understood as the source of exposure to market risk, and the accounts receivable (after netting and collateral) related to the obligors - as the source of exposure to credit risk.

For market risk, loss aggregation is the same as looking at the performance of a portfolio with all the price curves included in it. There can be gains and losses of such a portfolio,

[1]BP Oil International Limited, a company registered in England and Wales with the company number 322365 and VAT Number GB 243 5105 93 and whose registered office is Chertsey Road, Sunbury on Thames, Middlesex, TW16 7BP.

but for the sake of simplicity its performance is referred to as “total market loss“ (consistent with the term used for small-size model) - with gains understood as negative losses. The term “total market loss“ is also useful for risk measurement, which targets physical losses (not gains).

For credit risk, however, the translation of obligors’ equity returns into default states is necessary before aggregation by exposure can be done. This is where the additional data components are needed: obligors’ ratings and the credit migration matrix. Translation of equity returns into default states conforms to the methodology of CreditMetrics.

The exposures to obligors are assumed to have the form of accounts receivable. It is also assumed that changes of obligor credit conditions can produce losses only, in case of downgrade or default. No gains are made in case of reverse changes of obligors’ credit condition. Consistent with the terms used for market risk, losses incurred due to adverse changes in obligor credit conditions are referred to as “total credit loss“ (consistent with the term used for small-size model).

Loss aggregation ensures that the stationarity of the time series is not affected by changing exposures. This is because the dependence structure is calibrated to the history of returns and a specific combination of exposures. For the large-size model, individual risk factors show weak serial dependence, and for the purpose of this analysis are assumed to form an independent and identically distributed time series. With the exposures fixed, it translates directly to the stationarity of the bivariate historical data sample.

On the other hand, keeping the exposures fixed classifies the loss aggregation methodology as a static approach to aggregating risks. Some dynamics can be added by frequent rebuilding of the historical data sample by updating the exposures, or by narrowing the time horizon (down from one year). The annual time horizon is used for reporting purposes, and - if necessary - can be replaced by other appropriate time horizons (data frequencies).

For the large-size model, all market risk exposures and credit risk exposures were re-evaluated, so that they total to USD 1M each, with negative market risk exposures allowed. This re-evaluating operation preserves the original portfolio weights and does not affect the portfolio structure - only its size. It was made to ensure that for each combination of exposures the numerical results apply to comparable portfolios.

Another motivation for re-evaluating exposures is that making the total of market risk exposures and the total of credit risk exposures equal allows us to compare changes of VaR and ES implied by the dependence structure. However, in many cases the portfolio weights and the history of returns contributed to a significant disproportion between extreme values of total market loss and total credit loss.

It is worth bearing in mind that for a particular combination of exposures, the same portfolio weights are assumed to remain fixed throughout the time horizon selected for risk measurement, and relevant for the data frequency (as mentioned earlier, loss aggregation is a static approach to aggregating risks). In this report, the selected time horizon is one year.

Therefore, if significant portfolio weight is assigned to a seriously under-performing product (curve), the range of possible market losses becomes wide. The combination of weights can be such that holding a market portfolio of total value USD 1M with the exposures fixed throughout the year can result in maximum possible market loss of approximately USD 17M.

The history of total market loss (first random variable) and total credit loss (second random variable) provides a sample of a bivariate joint distribution. This translates into a simple bivariate case of dependence modelling. The high-dimensionality problem (very large number of risk factors) is thus avoided, and the dependence structure is applied to a pair of random variables.

For a model of this size, a comparison of all methodologies is hardly possible (risk factor aggregation is not accessible), but it is possible to compare the risk measurement process using loss aggregation and VaR aggregation. Although we don't have a reliable benchmark (unlike for the small-size model), some observations can be made by analysing of numerical examples.

3.2 Dependence structure for the large-size model

Initially, assuming that the total credit loss distribution for the large-size model can be treated as continuous, the standard copula types were tried. Four types of copulas were considered: 2 elliptical (Gaussian and t), and 2 Archimedean (Clayton and Frank). The copula types were selected so as to allow for both positive and negative dependence.

Estimation of the copula parameter(s) is non-parametric, always based on the empirical sample regardless of further assumptions on marginal distributions. All copulas except the t copula require the estimation of one parameter only. For the t copula, the estimation of its 2 parameters (ρ - the correlation coefficient, and ν - the degrees of freedom), can be reduced to a single-parameter estimation (see the algorithm presented in [McN05]). This technique is used here.

For the Archimedean copulas, their single parameter (θ) is related to Kendall's τ, and therefore to the standard correlation coefficient ρ. The construction of Archimedean copulas is not based on a given distribution type, and therefore their parameters are only indirectly related to the random variable correlation coefficients.

Fitting the right copula takes 2 steps: first, the copula parameters are fitted to each of the copula types. This is done using the historical sample of the total market loss and of the total credit loss. Then, the copulas with fitted parameters are compared in terms of accuracy of the dependence structure description.

Estimators of the copula parameters are the maximum likelihood estimators, based on copula density functions. To measure the accuracy of the dependence structure description, the AIC (Akaike Information Criterion) is used. This criterion takes into account both

the maximum likelihood and the number of parameters estimated. For details of copula parameter estimation and AIC construction see Appendix B. The copula which fits best is selected as the one describing the dependence.
Examples of results for copula fitting are presented in Table 3.1. Here, the optimal copula is the Clayton(0.1341) copula with the logarithmic likelihood of 1.6838 (the t copula has a higher logarithmic likelihood, but the fact it takes two parameters makes it suboptimal from the AIC point of view).

Standard copula parameters and AIC values for large-size model

Copula type	Gaussian(ρ)	t(ρ,ν)	Clayton(θ)	Frank(θ)
Parameter(s) value(s)	0.07758383	(0.0955923,6.719522)	0.1340836	0.5740442
Logarithmic likelihood	0.71445388	2.571719	1.6837739	1.1275052
AIC	0.57109225	-1.143438	-1.3675479	-0.2550104

TABLE 3.1: Copula Parameters and AIC Values for large-size model. *Each column corresponds to a selected standard copula type (Gaussian, t, Clayton, Frank). Copula names are provided with parameter names. The first row provides the parameter(s) values, the second - corresponding logarithmic likelihood, and the third - AIC values. Logarithmic likelihood values are marked red to facilitate comparison.*

Besides the fact that such a construction is questionable from the mathematical point of view, the logarithmic likelihood and AIC values returned by the optimisation algorithm were very low as compared to examples from literature, even considering the limited size of the sample - see, for example, [PB02].
For this very combination of exposures, the sample is dominated by relatively low values of total credit loss coinciding with values of total market loss slightly above their average. This translates into a correlation coefficient close to zero (0.0458, Kendall's τ 0.0610). However, rare high values of total credit loss coincide with values of total market loss well above the average. The observations including coinciding extreme loss values are too few to be reflected by the correlation coefficient, but should be captured by a properly fitted copula. If the copula indicates poor goodness-of-fit to the historical sample, it may be supposed that it omits the extreme values it should capture - and therefore produces a faulty dependence model in general.
Even though the distributions of both aggregate variables: total market loss and total credit loss, can be treated as continuous, standard copulas do not fit to the shape of the historical data sample. The low logarithmic likelihood and AIC values forced the search for an alternative dependence concept. The natural resource is the hypothetical copula used for modelling bivariate dependence for the small-size model. Indeed, provided it is mathematically justified, the hypothetical copula proves to have a much higher logarithmic

likelihood value (261.00), and the optimal AIC value (-518.0036) for the two parameters the simulation technique requires in its simplest form.

Historical and simulated losses for large-size model, simulation using the Clayton copula - scatter-plot

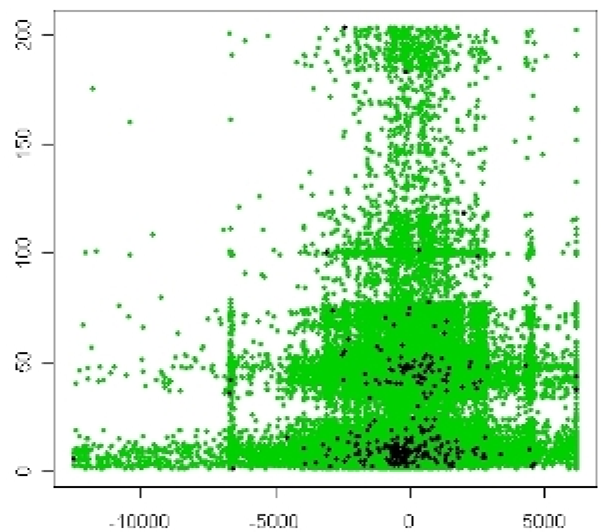

FIGURE 3.2: Historical and simulated losses for large-size model, simulation using the Clayton copula - scatter-plot. *Total market loss (X-axis) and total credit loss (Y-axis), in USD. Historical losses marked with black points, simulated losses - with green points.*

Historical and simulated losses for large-size model, simulation using the hypothetical copula - scatter-plot

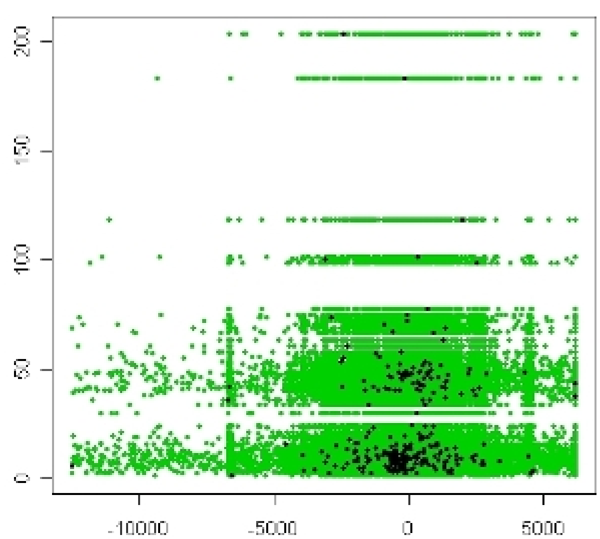

FIGURE 3.3: Historical and simulated losses for large-size model, simulation using the hypothetical copula - scatter-plot. *Total market loss (X-axis) and total credit loss (Y-axis) for large-size model, in USD. Historical losses marked with black points, simulated losses - with green points.*

The difference in simulation of joint values of total market loss and total credit loss, based on the Clayton copula and the hypothetical copula is presented on the scatter-plots above. The calibration involves the same historical sample, but the simulated points are located differently (see Appendix C for simulation of bivariate joint distribution with standard copulas and specified marginals).
As can be seen from scatterplots above, the total credit loss cumulates at certain levels also for the large-size model. The Clayton copula (Figure 3.2) produces an increased number of points between these levels, in a vertical area around the zero value of total market loss - forming a "chimney" shape. The hypothetical copula (Figure 3.3) produces an increased number of points exactly at cumulation levels and omits areas between them. The direct consequence is a substantially better copula fit.

3.3 Risk Measurement

Despite the change in the scatter-plot shape, the joint risk figures for both copulas in the above case were very close - the differences in values are within the statistical error limits (for the hypothetical copula the market VaR and the credit VaR are slightly higher, and the joint VaR - slightly lower than for the Clayton copula). The risk figures are presented in Table 3.4, matching the dependence structure defined above.

VaR and ES figures for large-size model
- the Clayton and the hypothetical copula

VaR (ES)	Loss aggregation - Clayton copula	Loss aggregation - hypothetical copula	VaR aggregation
Market VaR (ES)	6.1919 (6.1938)	6.1919 (6.1940)	6.1918 (6.1939)
Credit VaR (ES)	0.1986 (0.2009)	0.2032 (0.2032)	0.1975 (0.2003)
Aggregate VaR (ES)	6.2323 (6.2542)	6.2280 (6.2479)	6.2148 (6.2165)

TABLE 3.4: Market VaR (ES), credit VaR (ES), and aggregate VaR (ES) for large-size model, in USD M. *Figures calculated using the loss aggregation methodology with the Clayton copula as dependence structure (column 1), the hypothetical copula as dependence structure (column 2), and using the VaR aggregation methodology (column 3). Confidence level 99.9%.*

Also for other combinations of exposures, the risk figures resulting from the hypothetical copula and the optimal standard copula were quite the same. However, for standard copulas whose construction does not match the properties of the marginal distributions (mathematically), and whose AIC value is alarmingly low - risk figures could not be taken as reliable. The dependence structure given by the hypothetical copula was satisfactory from the goodness-of-fit criterion point of view, and by returning very similar risk figures, it justified the original ones. It is not excluded, however, that for a different historical data sample, the difference in joint risk figures resulting from different dependence structures could become more significant.

The loss aggregation methodology allows for considerable savings in economic capital, provided market regulations allow for market risk and credit risk to be treated jointly. Preliminary results indicate savings in VaR (ES) adding up to over 30% vs. the sum of VaRs (ESs).

The scale of savings vs. VaR aggregation reach approximately up to 15.5%, and depends heavily on the proportion of VaRs (ESs) calculated independently for each of the risk types. The proportion of VaRs (ESs) calculated independently depends, in turn, on the portfolio size and weights applied to the history of returns (see construction of the historical data sample). Since for the purpose of this analysis the portfolio sizes were unified, all disproportion is a consequence of portfolio weights.

Finally, the distribution shapes tend to adhere to some rules. As for the total market loss, it has a visibly fat right tail, which would not be captured if approximated by the normal distribution. It is captured, though, by the hypothetical copula and the empirical marginal distribution - even with some dose of exaggeration.

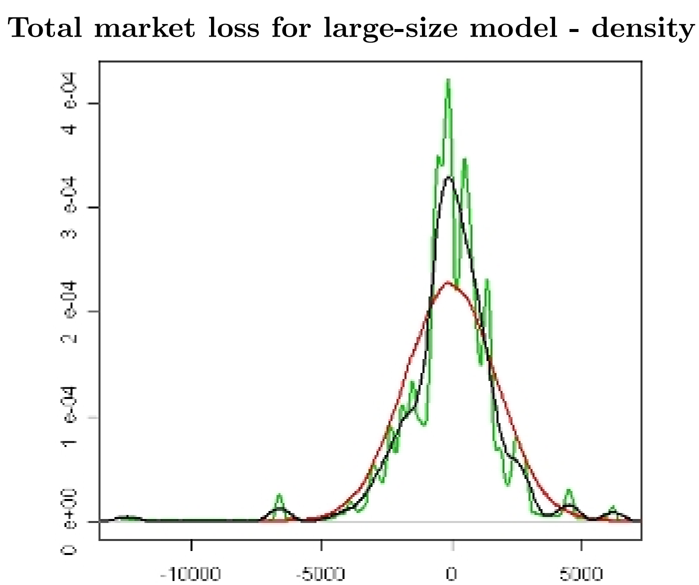

FIGURE 3.5: Total market loss for large-size model - density. *Green line - simulated density, black line - historical density, red line - shape of normally distributed density with the same*

mean and standard deviation as in the historical sample.

The total credit loss cumulates around certain levels, but this effect is not as distinctive as for the small-size model. This is related to the fact, that events of multiple defaults are rare and produce unique loss values in the historical sample. What is more, the cumulation levels influence all density shapes, making it strongly multi-modal. The total loss's local maxima are strictly related to the cumulation levels of total credit loss.

Total credit loss for large-size model - approximate density

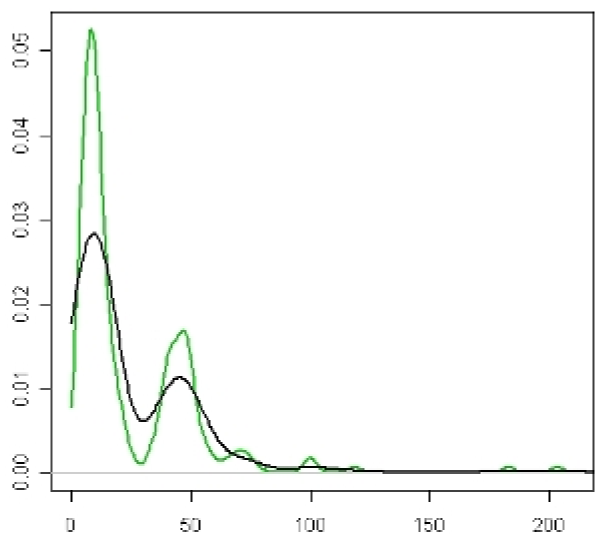

FIGURE 3.6: Total credit loss for large-size model - approximate density. *Green line - simulated density, black line - historical density.*

If the extreme values of total market loss and total credit loss are proportional, they both contribute to the value range of the total loss distribution. The total loss distribution shape is close to normal, however the approximation is weakest in the right tail area. The right tail of the total loss distribution is significantly fatter than the tail of the normal distribution - first because the shape is mainly inherited after the total market loss distribution, and second because the value range of the total credit loss distribution influences the value range proportionally.

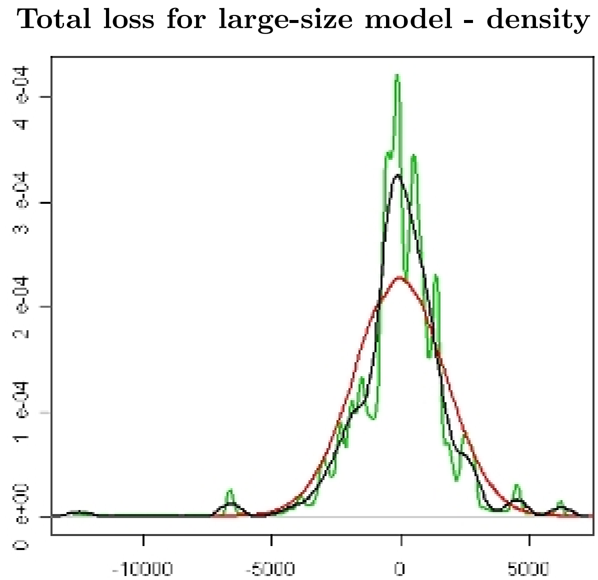

FIGURE 3.7: Total loss for large-size model - density. *Green line - simulated density, black line - historical density, red line - shape of normally distributed density with the same mean and standard deviation as the in the historical sample.*

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Appendix A: Decomposition of joint distributions into continuous marginal distributions and dependence structure given by copula

The dependence between real-valued random variables $X_1, ..., X_M$ is completely described by their joint distribution function:

$$F(x_1, ..., x_M) = P(X_1 \leq x_1, ..., X_M \leq x_M).$$

The concept of copulas stems from the idea of separating properties of marginal distributions (marginals) related to each variable individually, and properties of the dependence structure related to all variables. This concept allows for constructing multivariate distributions by combining arbitrary marginal distributions (known or estimated) and a particular type of dependence structure (implied by the copula).

DEFINITION A.1

Copula of random variables $(X_1, ..., X_M)$, *where* X_m *has the distribution function* F_m *for* $m = 1, ..., M$ *and* $F_1, ..., F_M$ *are continuous, is the joint distribution function of* $\mathbf{U} = (U_1, ..., U_M)$, *where* $U_m = F_m(X_m)$ *for* $m = 1, ..., M$.

As can be seen above, it is easier to speak about copulas from the perspective of (cumulative) distribution functions of particular random variables than from the perspective of the variables themselves. This is why the concept of copulas is inextricably bound with the probability integral transform $X \rightarrow F(X)$:

THEOREM A.2 (Fisher, 1931)

Let X be a random variable with distribution function F. Then, if F is continuous, the random variable $X \to F(X)$ is standard uniformly distributed, i.e $U = F(X) \sim \mathbf{U}(0,1)$, regardless the original distribution of X.

An immediate result is

COROLLARY A.3

Let F^{-1} be the quantile function of F, i.e.

$$F^{-1}(\alpha) = inf\{x : F(x) \geq \alpha\},\ \alpha \in (0,1).$$

Then for any standard-uniformly distributed $U \sim \mathbf{U}(0,1)$, $F^{-1}(U)$ has distribution function F.

The decomposition of a joint distribution into marginals and a copula relies on the following theorem:

THEOREM A.4 (Sklar, 1959)

Let $F_1, ..., F_M$ be continuous distribution functions of random variables $X_1, ..., X_M$ respectively, and let F be the joint distribution function of $\mathbf{X} = (X_1, ..., X_M)$. Then there exists a unique copula C such that:

$$F(x_1, ..., x_M) = C(F_1(x_1), ..., F_M(x_M)),$$

for all $(x_1, ..., x_M) \in \mathbb{R}^M$.
Conversely, if $F_1, ..., F_M$ are continuous distribution functions and C - the copula, then the function F defined as above is the M-variate joint distribution function with marginals $F_1, ..., F_M$.

The direct consequence is

COROLLARY A.5 (Building multivariate distributions)

Let C be a copula and $\mathbf{U} = (U_1, ..., U_M)$ have the distribution function C. If $F_1, ..., F_M$ are arbitrary continuous marginals and $F_1^{-1}, ..., F_M^{-1}$ their quantile functions, then $F_1^{-1}(U_1), ..., F_M^{-1}(U_M)$ has the copula C and marginals $F_1, ..., F_M$:

$$F(F_1^{-1}(U_1), ..., F_M^{-1}(U_M)) = C(u_1, ..., u_M).$$

The corollary above implies a 2-step procedure of fitting joint distribution to a vector of random variables. The first step is transforming random variables into values of their respective cumulative distribution functions by probability integral transform, and then fitting a copula to the transformed values. The second step is combining the fitted copula with marginal distributions for all random variables.

Appendix B: Estimation of the copula parameters and construction of AIC goodness-of-fit criterion

To estimate copula parameters, historical values of random variables (total market loss and total credit loss) are first transformed into values of their respective cumulative distribution functions via probability integral transform:

$$L_t^M \to F_M(L_t^M),$$
$$L_t^C \to F_C(L_t^C),$$

for every t, where L_t^M L_t^C are total market loss and total credit loss at time t respectively, and F_M and F_C are their cumulative distribution functions respectively.
F_M and F_C are approximated by the empirical distribution functions $\hat{F}_M$ and $\hat{F}_C$ respectively:

$$\hat{F}_M(x) = \tfrac{1}{T+1} \textstyle\sum_{t=1}^{T} \mathbf{1}_{\{L_t^M \leq x\}},$$
$$\hat{F}_C(x) = \tfrac{1}{T+1} \textstyle\sum_{t=1}^{T} \mathbf{1}_{\{L_t^C \leq x\}},$$

where T is the number of observations in time (length of the time series), and $\mathbf{1}_{\{L_t \leq x\}}$ is the indicator function of the loss value being less than x or equal.
Empirical distribution functions are used here to make the copula fit independent of assumptions on marginal distribution properties. This is so-called non-parametric estimation, as it does not involve any estimation of distribution parameters.
Copula parameters are fit based on transformed historical values. The following notation will be used:

$$\hat{F}_M(L^M) = \hat{U},$$
$$\hat{F}_C(L^C) = \hat{V}.$$

To simplify estimation of copula parameters, the marginal distributions will be standarised. This operation does not affect the description of dependence structure given by a copula, which is illustrated by the corollary below:

COROLLARY B.1 (Invariance)

If $X = (X_1, \ldots, X_n)$ has copula C and $T_1, \ldots, T_n$ are increasing continuous functions, then $T(X) = (T_1(X_1), ..., T_n(X_n))$ also has the copula C.

The above property means that the dependence structure implied by a copula is invariant under increasing and continuous transformations of marginals. Especially, it means that the copula $C(\theta, \mu, \Sigma)$ is identical to that of a $C(\theta, 0, P)$, where θ is the vector of copula parameters, μ is the vector of means, and P is the correlation matrix implied by the variance-covariance matrix Σ.
For each of the selected copula types, a maximum likelihood estimator of parameters is defined.

For the GAUSSIAN COPULA:

$$\ln L(P, \hat{U}, \hat{V}) = \sum_{t=1}^{T} \ln c_P^G(\hat{u}_t, \hat{v}_t),$$
$$c_P^G(u, v) = \frac{g_P(G^{-1}(u), G^{-1}(v))}{g(G^{-1}(u))g(G^{-1}(v))},$$

where g_P is the density of bivariate normal distribution with zero means and variance-covariance matrix P, and G is the distribution function of univariate standard normal distribution.

For the T COPULA:

$$\ln L(\nu, \Sigma, \hat{U}, \hat{V}) = \sum_{t=1}^{T} \ln c_{\nu,P}^t(\hat{u}_t, \hat{v}_t),$$
$$c_{\nu,P}^t(u, v) = \frac{t_{\nu,P}(T^{-1}(u), T^{-1}(v))}{t(T^{-1}(u))t(T^{-1}(v))},$$

where $t_{\nu,P}$ is the density of bivariate t distribution with zero means, variance-covariance matrix P, and degrees of freedom ν, and T is the distribution function of univariate standard t distribution with degrees of freedom ν.

For the CLAYTON COPULA:

$$\ln L(\theta, \hat{U}, \hat{V}) = \sum_{t=1}^{T} \ln c_{\theta}^{C}(\hat{u}_t, \hat{v}_t),$$
$$c_{\theta}^{C}(u, v) = (1+\theta)(uv)^{(-1-\theta)}[(u)^{-\theta} + (v)^{-\theta} - 1]^{(-\frac{1}{\theta}-2)},$$

where θ is the Clayton copula parameter and $\theta \in [-1, \infty)\backslash\{0\}$.

For the FRANK COPULA:

$$\ln L(\theta, \hat{U}, \hat{V}) = \sum_{t=1}^{T} \ln c_{\theta}^{F}(\hat{u}_t, \hat{u}_t),$$
$$\textit{defining } h(x) = \exp(-x\theta) - 1,$$
$$c_{\theta}^{F}(u, v) = -\theta h(1)(1 + h(u+v))/(h(u)h(v) + h(1))^2,$$

where θ is the Frank copula parameter and $\theta \in \mathbb{R}\backslash\{0\}$.

The Gaussian and the t copula are elliptical copulas, requiring the estimation of elements of matrix P. For a bivariate case, this is the correlation coefficient parameter between the two random variables. In addition, the t copula requires the estimation of the degrees of freedom parameter of the joint t distribution, after which the dependence structure is inherited. The additional parameter is denoted by ν.
It means that in the bivariate case, for the t copula there are two parameters to be estimated at once (ρ_{uv} - the single unknown element of the correlation matrix, and ν - the degrees of freedom). To facilitate this, a two-step estimation method is used. First, correlation coefficient ρ_{uv} between the two random variables is estimated. Next, the remaining parameter ν is estimated, keeping the correlation coefficient fixed.
To make this method more accurate, it is better to estimate correlation coefficients by means of Kendall's rank correlation, as the latter depends only on the copula and not on marginal distributions:

DEFINITION B.2 (Kendall's rank correlation)

Let $(X_1, \widetilde{X}_1)$ and $(X_2, \widetilde{X}_2)$ be two independent pairs of random variables having joint distribution function F. Then Kendall's rank correlation is given by:

$$\rho_{\tau} = E(sign((X_1 - X_2)(\widetilde{X}_1 - \widetilde{X}_2))).$$

For the Gaussian and the t copula, the following relation holds:

$$\rho_{\tau}(X_1, X_2) = \frac{2}{\pi \arcsin \rho},$$

where ρ is the standard Pearson's correlation coefficient between X_1 and X_2.
In order to arrive at the estimate of ρ, the estimator for ρ_τ is used first, and then the above relation is reversed to obtain ρ.

$$\widehat{\rho}_\tau(X_1, X_2) = (\begin{pmatrix} n \\ 2 \end{pmatrix})^{-1} \sum_{1<t_1,t_2\leq n} sign(x_{1,t_1} - x_{1,t_2})(x_{2,t_1} - x_{2,t_2}).$$

Being computationally easier, this method is also considered as giving estimates close to simultaneous estimation (see [McN05]).

The Clayton and Frank copulas are Archimedean copulas originating from a generator, not a joint distribution function of common type. Each of the copulas involves a single parameter (θ) measuring how strong the dependence between the two random variables is. There exists a link between these parameters and the Kendall's ρ_τ.

<u>For the CLAYTON COPULA:</u>

$$\rho_\tau(\theta) = \frac{1}{(\frac{2}{\theta}+1)};$$

<u>For the FRANK COPULA:</u>

$$\rho_\tau(\theta) = 1 - \frac{4}{\theta} + \frac{4}{\theta^2}\int_0^\theta \frac{t}{\exp(t)-1}dt;$$

In order to compare the copulas in terms of accuracy of dependence description, Akaike Information Criterion is used.

<u>DEFINITION B.3 (Construction of Akaike Information Criterion)</u>

Let $lnL(\theta, \mathbf{X})$ be the value of logarithmic likelihood, where θ is the vector of parameters and $\mathbf{X}$ the vector of arguments. Let k be the number of parameters. Then AIC is calculated according to the formula:

$$-2lnL(\theta, \mathbf{X}) + 2k;$$

The lower AIC, the better. For the purpose of this report, AIC can be understood as negative logarithmic likelihood, with a penalty for each additional parameter to be estimated.

Appendix C: Simulation of bivariate joint distribution with the best-fitting copula and specified marginals

The first step to risk measurement is to simulate values of bivariate joint distribution with the best-fitting copula and specified marginals.
For elliptical copulas - the Gaussian and the t copula - the simulation is performed according to the following procedure:

1. Simulate points $X = (x_1, x_2)$ from bivariate joint distribution (Gaussian or t, with respect to the selected copula type), with zero means and variance-covariance matrix P, and degrees of freedom ν if the t copula was selected;

2. Transform:

$$u = F(x_1) \text{ and } v = F(x_2),$$

where F is the distribution function of standard univariate Gaussian or t distribution, with respect to the selected copula type;

3. Apply the quantile functions of marginal distributions:

$$L_*^M = F_M^{-1}(u) \text{ and } L_*^C = F_C^{-1}(v),$$

where F_M^{-1} and F_C^{-1} are quantile functions of marginal distributions for total market loss and total credit loss respectively. The star index is used to identify simulated values;

For Archimedean copulas - the Frank and the Clayton copula - the simulation is performed in a different way, by means of the generator function:

1. Generate two independent uniformly ($\mathbf{U}(0,1)$) distributed random variables u and s;
2. Set:
$$w = \phi^{'}(-1)(\phi^{'}(u)/s),$$
where ϕ is the copula generator, ϕ^{-1} is the inverse function to the copula generator;
3. Set:
$$v = \phi^{[-1]}(\phi(w) - \phi(u));$$
where $\phi^{[-1]}$ is the quasi-inverse function to the copula generator;
4. Apply the quantile functions of marginal distributions:
$$L_*^M = F_M^{-1}(u) \text{ and } L_*^C = F_C^{-1}(v),$$
where F_M^{-1} and F_C^{-1} are quantile functions of marginal distributions for total market losses and total credit losses respectively;

<u>For the CLAYTON COPULA:</u>

$$\phi_\theta(u) = \tfrac{1}{\theta}(u^{-\theta} - 1),$$
$$v = [s^{-\theta/(\theta+1)}u^{-\theta} - u^{-\theta} + 1]^{-1/\theta};$$

<u>For the FRANK COPULA:</u>

$$\phi_\theta(u) = -\ln \tfrac{\exp(\theta u)-1}{\exp(\theta)-1},$$
$$v = -\tfrac{1}{\theta}[\ln(\tfrac{h(1)}{h(u)}(\tfrac{h(u)+1}{h(u)+1-h(u)h(w)} - 1)) + 1],$$
with $h(x) = \exp(-x\theta) - 1$, as in Appendix B.

Whatever the selected copula, the above procedures result in simulated values of jointly distributed total market loss and total credit loss, with the best-fitting copula and their respective marginals. In order to ensure high level of accuracy, 100,000 pairs are simulated (total market loss total credit loss).

Appendix D: Value at Risk and Expected Shortfall

DEFINITION D.1 (Value at Risk)

Given some confidence level $\alpha \in (0,1)$*, the Value at Risk (VaR) at the confidence level* α *is given by the smallest number* l *such that the probability that the loss* L *exceeds* l *is no larger than* $1-\alpha$*. Formally:*

$$VaR_{\alpha}(L) = inf\left\{l \in R, P(L > l) \geq 1-\alpha\right\};$$

This definition of VaR coincides with the definition of an α-quantile of the distribution of L, thus VaR itself can be understood as the α-quantile of distribution of L. If L is a normally distributed random variable, the quantiles of L can be expressed as scalars of standard deviation, and so can VaR_L.
Alpha, α is the standard notation for the VaR confidence level, i.e. the probability that losses do not exceed VaR_{α}. For example, $\alpha = 0.95$ means that losses do not exceed $\mathrm{VaR}_{0.95}$ with probability 0.95 and are higher than $\mathrm{VaR}_{0.95}$ with probability $1-\alpha = 0.05$.

DEFINITION D.2 (Expected Shortfall)

For continuous distributions, Expected Shortfall (ES) with respect to threshold VaR at confidence level $\alpha \in (0,1)$ *is defined as:*

$$ES_{\alpha} = E\left(L|L \geq VaR_{\alpha}\right) = \frac{E(L, L \geq VaR_{\alpha}(L))}{P(L \geq VaR_{\alpha})};$$

For arbitrary distributions (generalised version):

$$ES_{\alpha} = \frac{1}{1-\alpha}\left[E\left(L, L \geq VaR_{\alpha}(L) + q_{\alpha}\left(1-\alpha-P\left(L \geq VaR_{\alpha}\right)\right)\right)\right];$$

If the distribution is continuous, the second term on RHS of the generalised form is equal to 0, and both definitions are the same. The generalised form accounts for the fact that the probability mass can cumulate in single points (atoms).
ES can be understood as the average (expected) loss above the threshold given by VaR at confidence level α. ES in its generalised form is sub-additive and always more conservative than VaR.