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LONG-SHORT PORTFOLIO OPTIMISATION IN THE PRESENCE OF DISCRETE ASSET CHOICE CONSTRAINTS AND TWO RISK MEASURES

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Abstract

This paper considers long-short portfolio optimization in the presence of two risk measures: variance and Conditional Value at Risk (CVaR) and asset choice constraints of (i) buy, sell and holding thresholds (ii) cardinality restrictions on the number of stocks to be held in the portfolio. The mean-variance-CVaR model improves upon the classical mean-variance model by controlling both the variance and CVaR of the resulting return distribution. Our long-short extension to the mean-variance-CVaR model incorporates many financial institutions' practices in respect of the "short" decisions. We highlight that introducing short selling leads to superior choice of portfolios, with higher expected return and much lower risk exposures, as characterized by CVaR and variance. We further analyze the effects of applying buy and sell thresholds and cardinality restrictions on the number of stocks. Such constraints are of practical importance but make the efficient frontier discontinuous. When stocks' returns are represented as discrete random variables, the formulation leads to a Quadratic Mixed Integer Program (QMIP). We conclude that the long-short model with cardinality constraint is superior to the long only model even without cardinality constraint. The models are tested on real data drawn from the FTSE 100 index.

Subject classifications: Investment, Portfolio

Area of review: Finance

1. Introduction

Long-short portfolio selection strategies have generated much institutional interest in the last few years due to their importance in the weak market environment and their adoption in hedge funds. A long-short portfolio helps the investor to take advantage of the full array of securities and the full complement of investment insights by holding expected “winning securities” long and selling expected “losing securities” short. A long-short portfolio, by expanding the scope of the investment decisions, may lead to an improved performance when compared with a long only portfolio. However, the benefits are dependent to a large extent on the method of portfolio selection, which should use an integrated optimization of long-short positions - see Jacobs et al. (1996, 1997, 1998, 1999, 2006b).

Mean-risk models are still the most widely used approach in the practice of portfolio selection. With mean-risk models, return distributions are characterized and compared using two statistics: the expected return value and the value of a chosen risk measure. The risk measure chosen clearly plays an important role in decision making. The appropriate risk measures to be considered in the portfolio selection models have been a subject of debate. Variance was the first risk measure used in the mean-risk models (Markowitz 1952) and, in spite of criticism and many proposals of new risk measures (see for example Fishburn (1977), Yitzhaki (1982), Konno and Yamazaki (1991), Ogryczak and Ruszczyński (1999, 2001), Rockafellar and Uryasev (2000, 2002)), variance is still the most widely used measure of risk in the practice of portfolio selection. For regulatory and reporting purposes, risk measures concerned with the left tails of distributions (extremely unfavorable outcomes) are used. The most widely used risk measure for such purposes is Value-at-Risk (VaR). However, it is well known that VaR has undesirable theoretical properties (it is not sub additive, as shown, for example, in Tasche (2002) and thus fails to reward diversification). In addition, optimization of VaR leads to a non-convex NP-hard problem which is computationally intractable. In spite of a considerable amount of research, optimizing VaR is still an open problem (see for example Larsen et al. (2002), Leyffer and Pang (2005) and references therein). For these reasons, another risk measure concerned with the left tail, the Conditional Value-at-Risk (CVaR), is gaining more acceptance. CVaR has attractive theoretical properties: it controls the magnitude of losses beyond Value at Risk (VaR) and it is coherent (see for example Artzner et al. (1999), Acerbi and Tasche (2002), Tasche (2002), Pflug (2000), Rockafellar and Uryasev (2002)). For discrete random variables, represented by various outcomes under different scenarios, optimizing CVaR leads to a linear programming model of finite dimension (Rockafellar and Uryasev (2000, 2002)).

In an earlier work, Roman et al. (2007) proposed a model for portfolio selection that uses both variance and CVaR: the mean-variance-CVaR (MVC) model. This multi-objective model (maximize expected return, minimize variance, minimize CVaR) is transformed into a single objective optimization

problem, in which variance is minimized while suitable constraints are imposed on the expected return and CVaR level. The practical importance of this approach is twofold. Firstly, a solution obtained in this way has an intuitive appeal. For example, if the CVaR of a mean-variance efficient portfolio is considered to be unacceptably large, a constraint could be imposed on the CVaR level and a new portfolio obtained, which has a minimal variance under these conditions. The benefits of using both variance and CVaR, instead of just variance as the risk measure, have been underlined in Roman et al. (2007). Secondly, the problem is tractable from a computational point of view. In the case where the random variables under consideration are discrete and are described by their realizations under various scenarios, the long only portfolio choice is formulated as a quadratic programming (QP) model. Mitra et al. (2007) discuss the current state of art in solving QP portfolio planning problems.

The mean-variance-CVaR model is, however, limited in its application as it focuses on long only portfolios and hence on the selection of “winning securities”, ignoring a whole class of “losing” securities. In this paper we relax the long only constraint to incorporate the short selling decisions. Jacobs and Levy (1993, 1996, 1997) have tried to address many myths associated with long-short portfolio construction. Jacobs, Levy and Starer (1998, 1999) show that the true benefits of long-short portfolios are obtained only by an integrated optimization of long and short positions. The works of Jacobs, Levy and Markowitz (2005, 2006) show that the critical line algorithms used for long only portfolios can be used for calculating mean-variance efficient frontiers when factor or scenario model of covariance is considered, provided they satisfy certain conditions. They specify the conditions under which the covariance matrix can be diagonalised and discuss methods for representing the covariance matrix in the diagonal form. The resulting diagonal matrix is sparse and many algorithms have been developed for the fast processing of sparse quadratic programs (QP) - see Mitra et al. (2007). Konno et al. (2005) consider the long-short portfolio optimization problem under non-convex transaction cost for the mean-MAD framework. They approximate the non-convex function by linear underestimating functions and propose the use of branch and bound techniques as a computationally efficient way of processing such problems. They point out that, under declining market conditions, the long-short portfolios lead to better ex-ante and ex-post performance as compared to the long-only portfolios. Charpin and Lacaze (2002) and Kwan (2004) present analytical properties of the mean-variance long-short portfolios.

To capture the realism of portfolio planning, in this paper we extend the earlier work on the mean-variance-CVaR model (Roman et al. (2007)) to include (i) Short selling and constraints associated with short-selling in practice (ii) Discrete asset choice constraints.

These are summarized below:

(i) Short selling and constraints associated with short-selling in practice:

- a. Regulatory constraints: short selling decisions are influenced by regulatory requirements like Regulation T in US or internal lending policies of the broker.
- b. Budgetary constraints: the total amount invested should be less than or equal to 100% equity plus the amount borrowed.
- c. Investor's hedging strategies: short selling decisions are influenced by investor's strategy – for example, the market neutral strategy or an enhanced active equity strategy like the 120:20 strategy (Long: 120%, Short: 20%).

(ii) Discrete asset choice constraints:

- a. These may appear in two groups. Firstly, a minimum holding level may be defined. This requirement eliminates the unrealistically small positions that can otherwise be included in an optimized portfolio. Secondly, buy and sell thresholds may be defined: a minimum level below which an asset is not purchased or sold. (In a single period model where the portfolio is constructed from cash and thus no rebalancing is involved, it is enough to consider only buy and sell thresholds).
- b. Cardinality restrictions: investors and fund managers would like to specify the number of stocks to be held in their portfolio because of monitoring and control issues. They might also restrict the number of rebalancing buying or selling transactions.

Representing the mutual exclusivity of long and short positions with threshold restrictions on buy and sell positions and representing the cardinality restrictions require the introduction of binary variables. The incorporation of these additional features to the mean-variance-CVaR model changes the resulting formulation of the model from a QP to QMIP. The problem increases in size and becomes computationally complex. Jobst et al. (2001) use integer restart and re-optimization heuristics for QMIP when considering mean-variance models with cardinality, buy threshold and round lot constraints; they show that the mean-variance efficient frontier, in the presence of these constraints, becomes discontinuous. Bienstock (1996) and Lee and Mitchell (1997) use QMIP techniques to solve the portfolio selection problem with an upper limit on the size of the portfolios. Chang et al. (2000) use heuristic algorithms (genetic algorithm, tabu search and simulated annealing) to solve cardinality constrained problems with specified portfolio sizes. Linear programming based heuristics are used by Speranza

(1996), considering the negative semi MAD model with cardinality constraints. In our computational study, we apply the method described in Jobst et al. (2001) to the resulting QMIPs.

There has been a general opinion that “long-short portfolios are riskier”, due to the potentially unlimited loss implied by short selling. We show that, nevertheless, by imposing suitable CVaR constraints, the risk associated to the long-short portfolios can be not only controlled, but substantially reduced (as compared to the risk of the long only portfolios). In particular, the tail and the downside risk can be much reduced. Generally, by allowing short selling, much better portfolio choices can be obtained: higher expected return, less variance, less CVaR, higher values of well-known performance measures (Sharpe and Sortino ratios). However, the long-short efficient portfolios comprise more stocks than the long-only portfolios. Our computational study shows that, even with cardinality constraints on the number of stocks, the long-short portfolios are still better choices, having substantially less risk values and higher Sharpe / Sortino ratios, in comparison with their long only counterparts.

The rest of this paper is structured in the following way. In section 2, we discuss the mean-variance-CVaR long only model. The practice of short selling and the modeling details of the constraints and objectives in the presence of short selling are discussed in section 3; we also discuss the modeling details of applying threshold and cardinality constraints. In section 4, we present the computational results. A dataset drawn from FTSE 100 is used to evaluate the performance of our portfolio selection models. We further propose the use of integer restart heuristic, for overcoming the computational difficulties that are encountered in the models with discrete asset choice constraints of threshold and cardinality. In section 5 we set out the conclusions in a summarized form and discuss future directions of our research work.

2. The mean-variance-CVaR long only model

2.1. The mean-variance-CVaR model as a multi-objective problem

The portfolio selection problem with one investment period is an example of the general problem of choosing between random variables when larger outcomes are preferred. Decisions are required on the amount of capital to be invested in each of a number of available assets such that, at the end of the investment period, the return is as high as possible.

Consider a set of n assets, with asset $j \in \{1..n\}$ giving a return R_j at the end of the investment period. R_j is a random variable since the future price of the asset is not known. Let x_j be the proportion of capital invested in asset j ($x_j = w_j / w$, where w_j is the capital invested in asset j and w is the total

amount of the capital to be invested) and let $x = (x_1, \dots, x_n)$ be the portfolio resulting from this choice. This portfolio's return is the random variable $R_x = x_1 R_1 + \dots + x_n R_n$. In the simplest case and under no short selling constraints, the feasible set of decision vectors is

$$X = \{ (x_1, \dots, x_n) \mid \sum_{j=1}^n x_j = 1, x_j \geq 0, \forall j \in \{1, \dots, n\} \}$$

The problem of choosing between two feasible portfolios say x and y from this set X , becomes a choice between random variables R_x and R_y . In the classical mean-variance paradigm, the random variables R_x or R_y are characterized and compared by using two statistics: the expected value and the variance. In the mean-variance-CVaR paradigm (Roman et al. 2007), the random variables are characterized by using three statistics: the expected value, the variance and the CVaR at a specified confidence level $\alpha \in (0, 1)$. Thus, the non-dominated (efficient) solutions in the mean-variance-CVaR model are the Pareto efficient solutionsⁱ of a multi-objective problem in which the expected value is maximized while the variance and the CVaR are minimized:

$$(MVC): \max (E(R_x), -\sigma^2(R_x), -CVaR_\alpha(R_x))$$

Subject to: $x \in X$.

When plotting the efficient solutions in a mean-variance-CVaR space, a surface is obtained; we refer to this surface as “the efficient frontier” of the mean-variance-CVaR model.

We treat the returns of the assets as discrete random variables, and, using scenario generation or finite sampling of historical data, describe their realizations for T states of the world. For any $i \in \{1, \dots, T\}$, let state ω_i occur with probability p_i , $\sum_{i=1}^T p_i = 1$. Thus, the random returns are defined on a discrete probability space $\{\Omega, \mathcal{F}, P\}$ with $\Omega = \{\omega_1, \dots, \omega_T\}$, \mathcal{F} a σ -field and $P(\omega_i) = p_i$. Let r_{ij} be the return of asset j under scenario i , $i \in \{1, \dots, T\}$, $j \in \{1, \dots, n\}$. Thus, the random variable R_j representing the return of asset j is finitely distributed over $\{r_{1j}, \dots, r_{Tj}\}$ with probabilities p_1, \dots, p_T . The random variable R_x representing the return of portfolio $x = (x_1, \dots, x_n)$ is finitely distributed over $\{R_{x1}, \dots, R_{xT}\}$, where $R_{xi} = x_1 r_{i1} + \dots + x_n r_{in}$, $\forall i \in \{1, \dots, T\}$.

2.2. Representation of CVaR in the mean-variance-CVaR model

Let R_x be a random variable representing the return of a portfolio x over a given holding period and $\alpha \in (0, 1)$ a percentage ratio which represents a sample of “worst cases” for the outcomes of R_x (usually, $\alpha=0.01$ or $\alpha=0.05$).

The definition of CVaR at the specified level α is the mathematical transcription of the concept “average of losses in the worst A% of cases”ⁱⁱⁱ (Acerbi and Tasche 2002), where $A = 100\alpha$ % (for example $\alpha = 0.01$ implies $A=1\%$). A “loss” is usually defined as a negative outcome; thus, the loss associated with R_x is described by the random variable $-R_x$.

Formally, the Conditional Value-at-Risk at level α of R_x is defined as minus the mean of the α -tail distribution of R_x , where the α -tail distribution is obtained by taking the lower α part of the distribution of R_x (corresponding to extreme unfavorable outcomes) and rescaling its distribution function to span $[0,1]$:

$$CVaR_{\alpha}(R_x) = -\frac{1}{\alpha} \{ E(R_x 1_{\{R_x \leq q^{\alpha}(R_x)\}}) - q^{\alpha}(R_x) [P(R_x \leq q^{\alpha}(R_x)) - \alpha] \},$$

where q^{α} is an α -quantile of R_x , meaning that $P(R_x < r) \leq \alpha \leq P(R_x \leq r)$ (see Laurent 2003 for more details on α -quantiles),

and $1_{\{\text{Relation}\}} = 1$, if Relation is true

0, if Relation is false.

More details on CVaR can be found in Rockafellar and Uryasev (2000, 2002).

An important result, proved by Rockafellar and Uryasev (2000, 2002), and independently by Ogryczak (2002), is that the CVaR of a random variable R_x can be calculated by solving a convex optimization problem. Moreover, CVaR can be minimized over the set of feasible decision vectors. These results are summarized below:

Proposition 2.1 (CVaR calculation and optimization): Let R_x be a random variable depending on a decision vector x that belongs to a feasible set X , and $\alpha \in (0, 1)$. Consider the function:

$$F_{\alpha}(x, v) = \frac{1}{\alpha} E\{[-R_x + v]^+\} - v,$$

where $[u]^+ = \max\{u, 0\}$. Then:

(a) As a function of v , F_α is finite, continuous and convex;

(b)
$$CVaR_\alpha(R_x) = \min_{v \in R} F_\alpha(x, v), \quad \forall x \in X.$$

In addition, the set consisting of the values of v for which the minimum is attained, denoted by $A_\alpha(x)$, is a non-empty, closed and bounded interval (possibly formed by just one point).

(c) Minimizing $CVaR_\alpha$ with respect to $x \in X$ is equivalent to minimizing F_α with respect to $(x, v) \in X \times R$

$$\min_{x \in X} CVaR_\alpha(R_x) = \min_{(x, v) \in X \times R} F_\alpha(x, v).$$

In addition, a pair (x^*, v^*) minimizes the right hand side if and only if x^* minimizes the left hand side and $v^* \in A_\alpha(x^*)$.

(d) $CVaR_\alpha(R_x)$ is a convex function of x and $F_\alpha(x, v)$ is a convex function of (x, v) .

Thus, if the set X of feasible decision vectors is convex, minimizing CVaR is a convex optimization problem.

In the case when R_x is a discrete random variable, calculating and optimizing CVaR are linear programming problems. Suppose that R_x has T possible outcomes R_{x1}, \dots, R_{xT} with probabilities p_1, \dots, p_T . Then:

$$F_\alpha(x, v) = \frac{1}{\alpha} \sum_{i=1}^T p_i [v - R_{xi}]^+ - v.$$

For the portfolio selection problem, $R_{xi} = \sum_{j=1}^n x_j r_{ij}$ (r_{ij} is the return of asset j under scenario i),

hence:

$$F_\alpha(x, v) = \frac{1}{\alpha} \sum_{i=1}^T p_i [v - \sum_{j=1}^n x_j r_{ij}]^+ - v.$$

2.3. The mean-variance-CVaR long only scenario model

The efficient solutions of the (MVC) model can be obtained by solving the following single objective optimization problem (see Roman et al. (2007)):

$$(P_{L_MVC}): \min \sigma^2(x)$$

$$\text{Subject to: } F_\alpha(x, v) \leq z$$

$$E(x) \geq d$$

$$x \in X, v \in \mathbb{R}$$

where

$$X = \{ (x_1, \dots, x_n) \mid \sum_{j=1}^n x_j = 1, x_j \geq 0, \forall j \in \{1, \dots, n\} \}$$

For the portfolio selection problem, as presented in Section 2.1, consider T scenarios and n assets with

r_{ij} = the return of asset j under scenario i , for $i=1 \dots T$ and $j=1 \dots n$;

p_i = the probability of scenario i occurring, for $i=1 \dots T$;

μ_j = the expected return of asset j , $j=1 \dots n$;

σ_{jk} = the covariance between the returns of assets j and k , for $j, k=1 \dots n$.

The (MVC) model can be represented as the following QP:

$$\text{Min } \sum_{j=1}^n \sum_{k=1}^n x_j x_k \sigma_{jk}$$

Subject to:

$$\sum_{j=1}^n \mu_j x_j \geq d \quad (\text{d represents the desired level of expected return})$$

$$\frac{1}{\alpha} \sum_{i=1}^T p_i y_i - v \leq z \quad (\text{z represents the maximum level of CVaR})$$

$$y_i \geq v - \sum_{j=1}^n x_j r_{ij}, \quad \forall i \in \{1, \dots, T\}$$

$$y_i \geq 0, \quad \forall i \in \{1, \dots, T\}$$

$$\sum_{j=1}^n x_j = 1$$

$$x_j \geq 0 \quad \forall j \in \{1, \dots, n\}$$

The minimization is over $v, x_1, \dots, x_n, y_1, \dots, y_T$.

2.4. The efficient frontier of the mean-variance-CVaR model

The efficient solutions in the mean-variance-CVaR model have a level d for the expected value of return that must lie in the interval $[d_{\min}, d_{\max}]$. We define $d_{\min} = \max\{d_{\min\text{var}}, d_{\min\text{CVaR}}\}$, where $d_{\min\text{var}}$ and $d_{\min\text{CVaR}}$ are the expected returns of the minimum variance portfolio (mean-variance efficient) and minimum CVaR portfolio (mean-CVaR efficient) respectively. $d_{\min\text{var}}$ may be found as the optimal value of the variable d_0 in the problemⁱⁱⁱ:

$$\min \sigma^2(x)$$

$$\text{Subject to: } E(x) \geq d_0$$

$$x \in X, d_0 \in \mathbb{R}.$$

$d_{\min\text{CVaR}}$ may be found as the optimal value of the variable d_1 in the problem:

$$\min F_\alpha(x, v)$$

$$\text{Subject to: } E(x) \geq d_1$$

$$x \in X, v \in \mathbb{R}, d_1 \in \mathbb{R}.$$

To be more precise, $d_{\min\text{CVaR}}$ may be found as above only when the minimization of $F_\alpha(x, v)$ with respect to (x, v) over $X \times \mathbb{R}$ provides a unique optimal solution; nevertheless, the general case can be easily handled, see Roman et al. (2007).

We define d_{\max} as the maximum possible expected return^{iv}: the optimal value of the objective function in the problem:

$$\max E(x)$$

$$\text{Subject to: } x \in X.$$

Furthermore, for a specific $d^* \in [d_{\min}, d_{\max}]$, the level z of CVaR_α must lie in the interval $[z_{d^*, \min}, z_{d^*, \max}]$, where $z_{d^*, \min}$ is the best (minimum) CVaR_α level for the expected return d^* and $z_{d^*, \max}$ is the CVaR_α level of the (unique) portfolio that minimizes variance for the expected return d^* .

$z_{d^*, \min}$ is the optimal value of the objective function in the problem:

$$\text{Min } F_\alpha(x, v)$$

$$\text{Subject to: } E(x) \geq d^*$$

$$x \in X, v \in \mathbb{R}.$$

$z_{d^*,\max}$ may be found as the optimal value of the objective function in the problem:

$$\text{Min } F_\alpha(x^*,v)$$

Subject to: $v \in \mathbb{R}$,

where $x^* = (x^*_1, \dots, x^*_n)$ is the (unique) portfolio that minimizes variance for the mean return d^* .

The fact that the imposed limit z on CVaR_α is greater than or equal to $z_{d^*,\min}$ ensures that the problem (P_{L_MVC}) has a feasible solution, while z being less than or equal to $z_{d^*,\max}$ ensures that the constraint on CVaR in (P_{L_MVC}) is active. When solving problem (P_{L_MVC}) for a level of expected return equal to d^* and a CVaR level equal to $z_{d^*,\min}$, we obtain a mean-CVaR efficient portfolio; more precisely, the mean-CVaR efficient portfolio with the lowest variance for expected return d^* .

When solving problem (P_{L_MVC}) for a level of expected return equal to d^* and a CVaR level equal to $z_{d^*,\max}$, we obtain the mean-variance efficient portfolio with expected return d^* .

For a fixed level of expected return, the efficient solutions in the mean-variance-CVaR model form a curve when plotted in a CVaR-variance space, where the lower end of this curve is represented by the mean-CVaR efficient solution (with the highest variance) and the upper end is represented by the mean-variance efficient solution (with the highest CVaR). The other points of this curve are not efficient in either the mean-variance or the mean-CVaR model (Figure 2.1). However they may represent portfolios with a superior performance as compared to the mean-variance or mean-CVaR efficient portfolios.

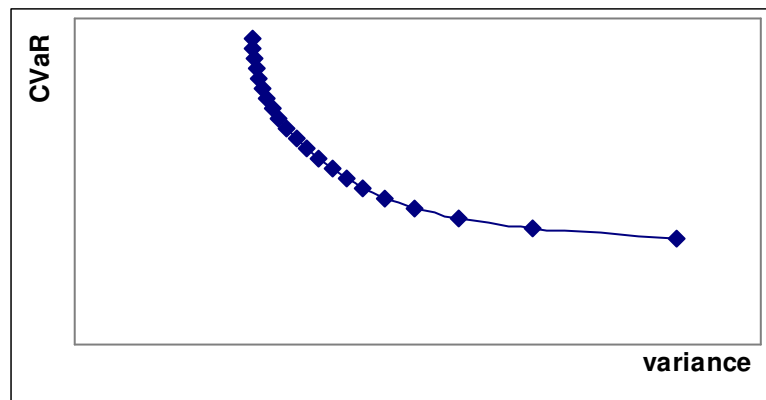


Fig. 2.1. The efficient solutions of the mean-variance-CVaR model, for a fixed level of expected value, plotted in a variance-CVaR space.

For the maximum level of expected return d_{\max} , this curve degenerates into just one point, with the coordinates equal to the variance and CVaR of the (only) efficient portfolio obtained for d_{\max} .

consisting of the asset with the highest expected return. Thus the MVC model does not exclude mean-variance and mean-CVaR efficient solutions. In addition it provides a whole range of solutions that are excluded with classical mean-variance models, although they could be superior choices.

3. The Mean-Variance-CVaR long-short model.

3.1. The importance of short selling

Most investors focus on the management of long portfolios and the selection of “winning” securities (the stocks whose prices would be rising in future). Yet the identification of winning securities ignores by definition a whole class of “losing” securities (the stocks whose prices would be going down in future).

The ability to sell short frees the investor to take advantage of stock price movement in both directions by making portfolio choice decisions: buy winning securities long, sell losing securities short. A long-short portfolio, by expanding the scope of the investor’s portfolio selection decisions, may lead to improved performance from security selection vis-à-vis a long-only portfolio. But the benefits of long-short are to a large extent dependent on the modeling methodology of portfolio construction. In the earlier practices, long-short portfolios were sometimes constructed by combining a long only portfolio, with a short only portfolio resulting in a long-plus-short portfolio, which is unlikely to produce an optimal long-short portfolio. Generally, the performance of a portfolio is assessed by comparing it to a benchmark, typically an equity index. In the long-plus-short portfolio, the long and short components are constructed relative to the benchmark index. Hence, the advantage offered by the flexibility to short is curtailed by the need to control risk by holding or shorting securities in index-like weights (see Jacobs et al. (1999)). Only an integrated optimization (selection of securities to be held long is determined simultaneously with selection of securities to be sold short) of long and short positions has the potential to maximize the value of investors’ insights. The benefits of the integrated optimization also include the freedom from the restrictions imposed by individual securities’ benchmark weights.

The benefits of long-short portfolio come with some increased costs; these are costs of borrowing the stocks for shorting, trading costs, management fees for the long-short portfolio. There is also theoretically increased risk due to potentially unlimited losses on the short positions. Nevertheless, these costs are within the explicit control of the investor and the long-short strategy is not necessarily costlier or riskier than the long-only strategy.

There are generally four different ways in which long-short investing strategies are implemented. These are briefly discussed below (see Jacobs and Levy 1993 and 2006 for a detailed discussion).

- a. Market neutral strategy – In this strategy, long and short positions are held in equal dollar balance all the times. As a result the net exposure to the equity market is eliminated and profits are made from the performance spread between stocks held long and stocks held short.
- b. Equitized strategy – In this strategy, in addition to holding long and short positions in equal dollar balance, a permanent stock index futures overlay is added in an amount equal to the invested capital. Profits are made from the long-short spread in addition to profits or losses resulting from equity market’s rise or fall.
- c. Hedge strategy – This strategy also recommends holding long and short positions in equal dollar balance but has a variable equity market exposure depending upon the market outlook, achieved using stock index futures. Profits are made from long-short spread in addition to profits or losses resulting from changing stock index futures position.
- d. Enhanced active equity strategy – This strategy seeks to improve the performance of actively managed long-only portfolios by allowing for short selling and reinvestment of short sale proceeds in incremental long positions. 120-20 strategy is a very popular example of this type of strategy.

In the first three strategies discussed above the investor may be a customer of prime broker, holding a regular margin account which is governed by the margin regulations like Regulation T. In the enhanced active equity strategy the investor is counterparty in the stock loan transaction; he holds a stock loan account and his amount of borrowing is subject to broker’s own internal lending policies, not to regulatory restrictions.

3.2. Formulation of the long-short scenario model

We represent the asset-universe by $2n$ non-negative variables (n variables for long positions and other n variables for short positions, for each of the n assets). Thus the feasible set of portfolio decisions is defined as:

$$X^{LS} = \{(x_{L1}, \dots, x_{Ln}, x_{S1}, \dots, x_{Sn}) \mid x_{Lj}, x_{Sj} \text{ subject to constraints elaborated in Table 3.1, } \forall j \in 1 \dots n\}$$

The procedure described in section 2.4 for computing the long only efficient frontier (applicable for $x \in X$) also applies to $x \in X^{LS}$, that is, for the long-short model.

3.2.1. Short selling constraints in practice

In Tables 3.1 - 3.4 we present and discuss the long-short extension of the mean-variance-CVaR model. For detailed discussions of the long-short models also see Jacobs et al. (2005, 2006), Konno et al. (2005), Charpin and Lacaze (2002) and Kwan (2004).

<p>Index sets :</p> <p>$j = 1 \dots n$ denotes the asset universe</p> <p>$k = 1 \dots m$ denotes different regulatory (or enhanced active) constraints</p> <p>Variables :</p> <p>x_{Lj} represents the portfolio weight in long position in the j-th stock</p> <p>x_{Sj} represents the portfolio weight in short position in the j-th stock</p> <p>x_c represents the cash balance.</p> <p>x_b represents the amount borrowed.</p> <p>δ_{Lj}, δ_{Sj} binary variables associated with long and short positions</p> <p>$x_{Lj} \geq 0, x_{Sj} \geq 0, \forall j \in 1 \dots n$</p> <p>$x_c \geq 0, x_b \geq 0$</p> <p>Parameters :</p> <p>p_{kj} represents the coefficient for the long position on stock j in the k-th constraint.</p> <p>q_{kj} represents the coefficient for the short position on stock j in the k-th constraint.</p> <p>b_k represents the k-th constraint limit value</p> <p>L_{lower} represents the lower bound for the each long position.</p> <p>L_{upper} represents the upper bound for the each long position.</p> <p>S_{lower} represents the lower bound for the each short position.</p> <p>S_{upper} represents the upper bound for the each short position.</p> <p>h_j represents the percentage of interests earned on short position deposits for stock j, which is passed back to the investor.</p> <p>r_c represents the interest rate on cash or collateral</p> <p>s is a parameter defining investor's strategy.</p>	
Constraint/Model details	Algebraic representation/Comments
1. Regulatory restriction (e.g. Regulation T) or enhanced active	$\sum_{j=1}^n (p_{kj}x_{Lj} + q_{kj}x_{Sj}) \leq b_k; \quad \forall k = 1 \dots m;$ <p>Regulation T type of constraint may be represented as</p>

<p>equity type of constraints</p>	$\sum_{j=1}^n (px_{Lj} + qx_{Sj}) \leq 1; \text{ For Regulation T, } p = q = 0.5$ <p>In our investigation we consider an alternative enhanced active equity 120:20 strategy which is modeled as:</p> $\sum_{j=1}^n x_{Lj} = 1.2$ $\sum_{j=1}^n x_{Sj} = 0.2$
<p>2. Total value of Long – Total value of short, be close to some investor specified value, s</p>	$\left \sum_{j=1}^n (x_{Lj} - x_{Sj}) - s \right \leq \tau;$ <p>This is equivalently modeled as two sided range constraint:</p> $(s - \tau) \leq \sum_{j=1}^n (x_{Lj} - x_{Sj}) \leq (s + \tau)$ <p>We consider full market exposure, that is $s = 1$ and zero tolerance, that is $\tau=0$.</p>
<p>3. Variable lower and upper bounds on individual long and short positions.</p>	<p>This is modeled as:</p> $\delta_{Lj} * L_{lower} \leq x_{Lj} \leq \delta_{Lj} * L_{upper}, \quad \forall j \in 1 \dots n$ $\delta_{Sj} * S_{lower} \leq x_{Sj} \leq \delta_{Sj} * S_{upper}, \quad \forall j \in 1 \dots n$ <p>We consider: $L_{lower}=S_{lower}=0.0001$ and $L_{upper}=S_{upper}=1$.</p>
<p>4. Budget constraint</p>	$\sum_{j=1}^n x_{Lj} + x_c - x_b \leq 1;$
<p>5. Short rebate</p>	<p>When a security is sold short, the proceeds of the short sale plus some extra margin or long positions worth the amount shorted are deposited with the broker as collateral. This cash or collateral may earn some interest, a part of which is passed back to the investor.</p> <p>We consider enhanced active equity 120:20 strategy under which the broker would generally not provide any interest on the collateral. Hence interest rate on collateral $r_c = 0\%$</p>
<p>6. Mutual exclusivity of</p>	$x_{Lj}x_{Sj} = 0, \quad \forall j \in 1 \dots n$

long and short position on the same stock (this is modeled together with 3.)	This is equivalently modeled using associated binary variables as: $\delta_{L_j} + \delta_{S_j} \leq 1; \quad \forall j \in 1 \dots n$
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Table 3.1. The long-short extension of the MVC model: short selling constraints.

3.2.2. Objective functions under short selling: mean, variance, CVaR

The long short extension of the (MVC) model naturally has M: Expected return (Mean), V: Variance and C: CVaR as three objective functions. Where relevant, an objective function is replaced by a constraint with appropriate range, that is, lower and upper limits on its realizable values. Similarly to the long only MVC model, the variance of the portfolio return is minimized while suitable constraints are imposed on CVaR and on the expected return.

The expected return / Mean: The expected return for a long-short model includes the returns from long and short positions and contains an additional term for returns obtained as short rebate.

The constraint on the expected return	$\sum_{j=1}^n (\mu_j x_{L_j} - \mu_j x_{S_j} + r_{cash} x_{S_j} h_j) \geq d;$ <p>d = the desired expected return, μ_j = the expected return of the j-th stock</p>
---------------------------------------	---

Table 3.2. The long-short extension of the MVC model: the constraint on the expected return.

The variance: If Q denotes the covariance matrix between n assets for the long only model, for the long-short model the covariance matrix is as shown below:

$$\begin{pmatrix} Q & -Q \\ -Q & Q \end{pmatrix}$$

Variance defines the objective function (to be minimized)	$\min \sum_{j=1}^n \sum_{k=1}^n (\sigma_{jk} x_{L_j} x_{L_k} + \sigma_{jk} x_{S_j} x_{S_k} - 2\sigma_{jk} x_{L_j} x_{S_k})$ <p>Or equivalently,</p> $\min \sum_{j=1}^n \sum_{k=1}^n (x_{L_j} - x_{S_j}) \sigma_{jk} (x_{L_k} - x_{S_k})$ <p>σ_{jk} = covariance between stock j and k.</p>
---	--

Table 3.3. The long-short extension of the MVC model: the objective function.

CVaR: In the presence of short selling, the portfolio return incorporates the returns from short positions and the short rebate. The CVaR of the portfolio is thus modeled as:

The constraint on CVaR	$y_i \geq v - \sum_{j=1}^n (r_{ij}x_{Lj} - r_{ij}x_{Sj} + r_c x_{Sj} h_j)$ $\frac{1}{\alpha} \sum_{i=1}^T p_i y_i - v \leq z$ $y_i \geq 0$
------------------------	--

Table 3.4. The long-short extension of the MVC model: the constraint on CVaR.

The complete formulation of the long-short MVC model with enhanced active equity strategy 120:20 is set out in Appendix A. In Appendix B we formulate an alternative long-short MVC model, under margin trading and Regulation T type short selling constraints.

3.3 Introducing threshold and cardinality constraints

Discrete constraints of threshold and cardinality, representing practical trading requirements, introduce discontinuities into the otherwise continuous efficient frontier (see Jobst et al. (2001)). When the mean return and two risk measures are considered simultaneously in the portfolio selection model, the efficient frontier is a surface. In the presence of discrete asset choice constraints, this surface becomes discontinuous. Applying the analysis presented in Jobst et al. (2001) we argue, that with the number of stocks to be held in the portfolio fixed, the ‘Pareto optimal solutions’ of the MVC model (long or long-short) may exclude a few expected return levels (lying within the expected return range for the efficient portfolios) or may cause discontinuities in the CVaR-variance plot (at a given expected return level).

The threshold constraints go together with the modeling requirements of long-short portfolios in our representation. Two binary variables δ_{Lj} and δ_{Sj} are used together with the lower and upper bounds on individual long and short positions (buy and sell) correspondingly. The modeling aspect of these constraints has already been shown in Table 3.1, item 3.

In practice, because of monitoring and control issues, an investor would generally impose cardinality restrictions (invest in a limited number of assets). In the case of the long only model, such restrictions are represented by summing the binary variables associated to the portfolio weights. For the (MVC) long-short model we can apply cardinality restrictions in two ways:

a. Cardinality on the long only positions of the portfolio (modeled as $\sum_{j=1}^n \delta_{Lj} = k_l$, where k_l is the desired number of long positions).

b. Cardinality on the long-short positions taken together (modeled as $\sum_{j=1}^n (\delta_{Lj} + \delta_{Sj}) = k_{ls}$, where k_{ls} is the desired number of positions).

4 Computational study

4.1 Scope and dataset description

In this section we investigate the performance of the MVC long-short model as compared to the MVC long only model. We also investigate the effects of applying cardinality constraints for both long only and long-short models. Thus, four generic models are considered. The first two are the long only and long short MVC models where no cardinality constraints are considered. In the last two models these constraints are imposed as additional restrictions on portfolio choice.

A dataset, drawn from the FTSE 100 index, was used for this analysis. The returns of the 76 stocks that belonged to the index throughout the period January 1993 - December 2003 were considered (for each of the remaining 24 stocks data there is at least one missing data item in the specified period). The dataset consists of monthly returns and has 132 time periods, considered as equally probable scenarios ($n=76$, $T=132$). We use CVaR at 0.01 confidence level. For the out-of sample analysis, the behavior of the portfolios obtained was examined over the eighteen months following the date of selection (January 2004 - June 2005). The models were written in AMPL (Fourer et al. 1993) and processed using the CPLEX 10.1 optimisation solver. The covariance matrix is computed from historical data. The short selling parameter values (interest on collateral $r_c=0\%$, the 120:20 strategy) were already described in Table 3.1.

4.2 Mean-variance-CVaR long and long-short models without cardinality constraints

4.2.1 In-sample analysis

We first note that, by allowing short selling, higher expected returns may be achieved. The efficient portfolios in the MVC long only model have expected returns in the interval [0.0093, 0.0331], while the efficient portfolios in the MVC long short model have expected returns in the interval [0.0104, 0.0390].

We consider four return levels attainable by both long and long-short portfolios: $d_1=0.0140$, $d_2=0.0188$, $d_3=0.0236$, $d_4=0.0283$. For each level of expected return d_i , with $i=1\dots 4$, we determine $z_{d_i,\min}$: the minimum level of CVaR (corresponding to the mean-CVaR efficient portfolio) and $z_{d_i,\max}$: the maximum level of CVaR (the CVaR of the mean-variance efficient portfolio with expected return d_i) and, between them, another 3 equally spaced levels of CVaR. Thus, the interval $[z_{d_i,\min}, z_{d_i,\max}]$ for CVaR is divided into 4 equal parts. For a specific level of expected return, when solving the mean-variance-CVaR model with these CVaR levels, we obtain 5 portfolios, denoted by: P_{CVaR} , $P_{1/4CVaR}$, $P_{1/2CVaR}$, $P_{3/4CVaR}$ and P_{var} respectively. Thus, P_{CVaR} is the mean-CVaR efficient portfolio (with the lowest variance, for the specified expected return) and P_{var} is the (unique) mean-variance efficient portfolio for the specified expected return (See Fig. 4.1).

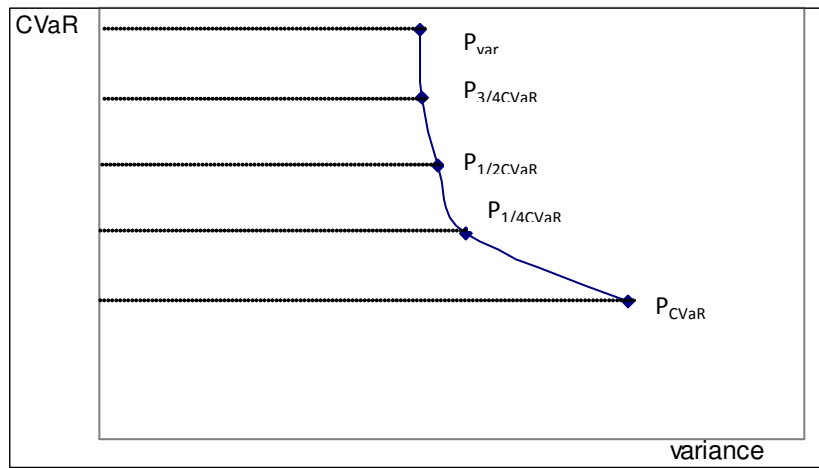


Fig. 4.1. The efficient frontier for a fixed level of expected return in a CVaR-variance space. The interval for CVaR is divided into 4 equal parts.

For each level of expected return d_1, d_2, d_3, d_4 , we consider thus five long-only MVC efficient portfolios and five long-short MVC efficient portfolios, as described above. We investigate the composition and the return distributions of these portfolios.

In the case of the long-short model, much better (lower) CVaR levels can be achieved: for a specific expected return level d^* , the feasible interval $[z_{d^*,\min}, z_{d^*,\max}]$ for CVaR is shifted “downwards” (see Fig. 4.2). Moreover, when solving the corresponding optimization problems, better (lower) variances are obtained in the case of the long-short model. From Fig. 4.2 it can be seen that the CVaR-variance plot for the efficient portfolios of the long-short model shifts left and downwards, which implies better (lower) risk values for the long-short portfolios. This pattern is repeated for all return levels, as it can be seen from Table 4.1.

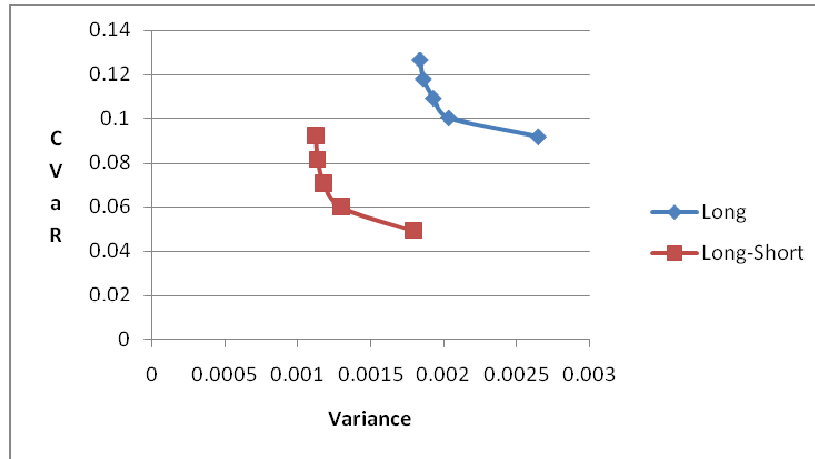


Fig. 4.2. In-sample analysis, CVaR-variance plot, for the expected return level of 0.0188

Expected return	Long			Long-Short			
	CVaR	Variance	Number of stocks	CVaR	Variance	Number of stocks	
						Long	Short
0.014034	0.0706	0.0019	7	0.0402	0.0013	16	3
	0.0791	0.0013	17	0.0498	0.0009	25	7
	0.0876	0.0012	20	0.0593	0.0008	23	6
	0.0961	0.0012	22	0.0689	0.0008	25	6
	0.1046	0.0011	21	0.0785	0.0008	26	7
0.018801	0.0919	0.0026	7	0.0490	0.0018	11	1
	0.1005	0.0020	11	0.0598	0.0013	19	3
	0.1091	0.0019	14	0.0706	0.0012	21	4
	0.1177	0.0019	13	0.0814	0.0011	21	5
	0.1263	0.0018	13	0.0922	0.0011	23	7
0.023567	0.1225	0.0051	6	0.0709	0.0028	8	1
	0.1314	0.0037	8	0.0821	0.0022	12	3
	0.1404	0.0035	9	0.0933	0.0020	11	4
	0.1493	0.0034	8	0.1045	0.0019	11	4
	0.1583	0.0034	7	0.1157	0.0019	12	4
0.028334	0.1913	0.0083	4	0.1046	0.0046	6	2
	0.1937	0.0079	5	0.1144	0.0040	10	3
	0.1961	0.0077	5	0.1243	0.0038	8	4
	0.1985	0.0076	6	0.1341	0.0037	8	4
	0.2009	0.0076	6	0.1440	0.0036	9	4

Table 4.1: In-sample analysis, CVaR – variance and cardinality comparison

Long-short portfolios are thus less “risky”, but they comprise many more stocks, as it can be seen from Table 4.1. The mean-CVaR efficient portfolios have the smallest number of stocks; the number of stocks increases when increasing the CVaR level. As the expected return level is increased, the number of stocks required to obtain efficient portfolios decreases both for long and for long-short models (see Table 4.1).

We also evaluated the portfolios described above by calculating the values of two well-established performance measures: the Sharpe and the Sortino ratios; they are displayed in Table 4.2.

The Sharpe ratio simply expresses the ratio between the excess rate of return and the standard deviation (the excess rate of return is the expected rate of return minus the risk free rate of return; we considered an annual risk free rate of 4.5% and thus a monthly risk free rate of 0.375%.) Thus, the Sharpe ratio measures the rate of return in excess of the risk-free rate of return per unit of standard deviation.

Expected return	Long			Long-Short		
	CVaR	Sharpe ratio	Sortino ratio	CVaR	Sharpe ratio	Sortino ratio
0.014034	0.0706	0.2376	0.6313	0.0402	0.2844	0.8466
	0.0791	0.2896	0.6880	0.0498	0.3459	0.9668
	0.0876	0.2985	0.6987	0.0593	0.3640	0.9427
	0.0961	0.3026	0.6978	0.0689	0.3699	0.9265
	0.1046	0.3038	0.6889	0.0785	0.3716	0.9085
0.018801	0.0919	0.2924	0.6969	0.0490	0.3551	1.0249
	0.1005	0.3336	0.7856	0.0598	0.4176	1.1272
	0.1091	0.3427	0.8021	0.0706	0.4383	1.1189
	0.1177	0.3489	0.8067	0.0814	0.4466	1.1096
	0.1263	0.3511	0.8033	0.0922	0.4491	1.0855
0.023567	0.1225	0.2778	0.6347	0.0709	0.3726	1.0172
	0.1314	0.3242	0.7197	0.0821	0.4211	1.0858
	0.1404	0.3337	0.7366	0.0933	0.4384	1.1243
	0.1493	0.3395	0.7468	0.1045	0.4491	1.1315
	0.1583	0.3415	0.7498	0.1157	0.4526	1.1119
0.028334	0.1913	0.2700	0.5876	0.1046	0.3608	0.8706
	0.1937	0.2766	0.6010	0.1144	0.3898	0.9322
	0.1961	0.2803	0.6026	0.1243	0.4004	0.9490
	0.1985	0.2814	0.6001	0.1341	0.4063	0.9491
	0.2009	0.2815	0.5993	0.1440	0.4081	0.9405

Table 4.2: In-sample analysis: performance indices for long and long-short portfolios

With the Sortino ratio, only returns falling below a target return T (a minimum acceptable return, specified by the investor) are penalized. (We considered $T=0$, meaning we only penalized the negative returns.) The Sortino ratio measures the rate of return in excess of the target return per unit of downside risk. The downside risk is the square root of the target semi-variance (i.e. the average of the squared negative deviations from the target return). The definition of the Sortino ratio and a discussion of downside risk may be found in Sortino and Satchell (2001).

As it may be seen from Table 4.2, the values of these performance measures are much higher in the case of long-short portfolios. This was to be expected in the case of the Sharpe ratio, since the long-short portfolios have lower variances. The fact that the Sortino ratios are also much higher for long-short portfolios shows that, by allowing short-selling, the downside risk (as measured by the target semi-variance) can be substantially reduced.

4.2.2 Out-of-sample analysis

We analyze the performance of the portfolio decisions described in Section 4.2.1 for the next 18 time periods (Jan 2004 to June 2005) following the date of selection. We compute the statistics of the 18 out-of-sample returns; they are displayed in Table 4.2, for the portfolios with an in-sample expected return of 0.0188 and the in-sample CVaR levels specified in Section 4.2.1 (as previously explained, we cannot use the same in-sample CVaR for long only and long-short portfolios, since the CVaR levels of the long-short portfolios are much lower).

As seen from Table 4.3, the long-short portfolios have much better statistics as compared to the long only counterpart (higher mean, higher median, lower standard deviation and higher minimum). In Figure 4.3, we plot the out-of-sample cumulative returns of the same portfolios (with the in-sample expected return of 0.0188). The better out-of-sample performance of the long-short portfolios is obvious.

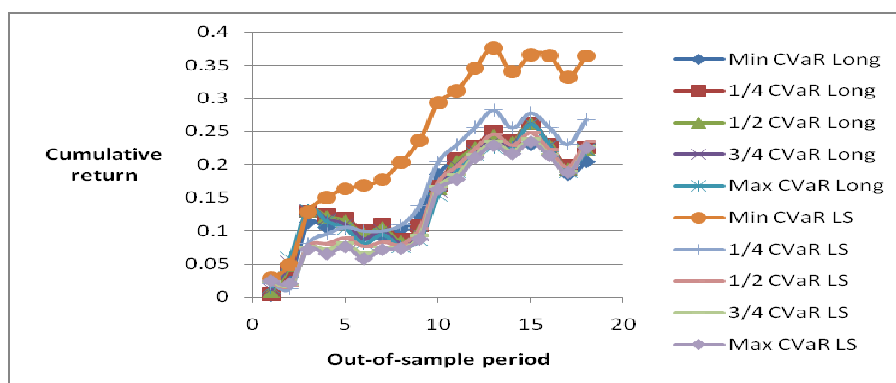


Fig. 4.3. Portfolio out-of-sample cumulative returns for 01-01-2004 to 06-01-2005 (In-sample portfolios constructed at expected return of 0.0188 and at different CVaR levels)

Expected return	Identifier	In-sample CVaR	Mean	Median	Std Dev	Min	Max
			Long				
0.0188	Min CVaR-Long	0.0919	0.0113	0.0102	0.0287	-0.0425	0.0818
	¼ CVaR-Long	0.1005	0.0125	0.0138	0.0319	-0.0326	0.0875
	½ CVaR-Long	0.1091	0.0123	0.0131	0.0322	-0.0361	0.0827
	¾ CVaR-Long	0.1177	0.0124	0.0124	0.0322	-0.0382	0.0780
	Max CVaR-Long	0.1263	0.0123	0.0123	0.0326	-0.0401	0.0729
	Identifier	In-sample CVaR	Long-Short				
	Min CVaR-LS	0.0490	0.0203	0.0239	0.0272	-0.0362	0.0800
	¼ CVaR-LS	0.0598	0.0149	0.0169	0.0272	-0.0269	0.0683
	½ CVaR-LS	0.0706	0.0131	0.0134	0.0267	-0.0265	0.0701
	¾ CVaR-LS	0.0814	0.0126	0.0141	0.0264	-0.0262	0.0735
	Max CVaR-LS	0.0922	0.0125	0.0140	0.0260	-0.0257	0.0750

Table 4.3. Long vs. Long-Short: Statistics of the 18 out-of-sample returns (portfolios with an in-sample expected return of 0.0188 and different CVaR levels)

4.3 Mean-variance-CVaR long and long-short models with cardinality constraints

4.3.1 Integer restart heuristics

When adding cardinality constraints to the long-short MVC model, the problem becomes computationally challenging. In order to obtain solutions in acceptable time we may restrict the complete search in the branch and bound tree. However, if we restrict the search and obtain sub-optimal solutions, our experiments show that we are likely to lose the ‘Pareto efficient’ property of the efficient frontier. Hence, we adopt the following scheme of computing the efficient frontier, used also in Jobst et al. (2001). For any expected return level d^* , we vary the CVaR level in the corresponding optimization problem from the minimum level $z_{d^*,\min}$ to the maximum level $z_{d^*,\max}$. In this sequence, we use the previous integer solution as the starting point for the solution search at the next CVaR level. Given the previous solution is feasible (and may be optimal), this solution is automatically a feasible solution for the current optimization problem, since the constraint on CVaR is relaxed. This has the effect that we obtain an efficient frontier (discontinuous) which is optimal (if all problems are solved to optimality) or suboptimal

(if the algorithm is terminated at a feasible solution). However, the frontier we generate cannot contain inefficient points, or, equivalently, dominated solutions, as we either stay at the previous solution or we improve upon it. Our experiments show that the use of integer restart heuristics, even in cases where we do not restrict the search, helps in reducing the solution time considerably.

4.3.2 In-sample analysis

In section 4.2.1, we pointed out that long-short portfolios comprise many more stocks than the long only portfolios. To support our argument that the improvement achieved in risk values by allowing short-selling is not due only to the “diversification” effect (increased number of stocks in the portfolio), we impose the same cardinality restrictions on both long and long-short models and compare the solutions obtained.

For the long-short model we can apply cardinality constraints in two ways, as discussed in Section 3.3: on long positions only or on long-short positions together. Thus, we consider three MVC models with cardinality constraints: the long only model, the long-short with cardinality constraints on the long positions only and the long-short with cardinality constraints on the total number of positions.

We select efficient portfolios of these models with the same cardinality and the same expected return and analyze their return distributions.

We consider three expected return levels attainable by both long and long-short portfolios: $d_1=0.0140$, $d_2=0.0188$, $d_3=0.0236$. For each such expected return level, we consider two cardinalities (cardinalities 15 and 10 for d_1 , cardinalities 10 and 8 for d_2 and cardinalities 6 and 5 for d_3 .) While imposing a specified expected return and cardinality restrictions, the CVaR range of the MVC efficient portfolios varies with the specified cardinality imposed. (Thus, we do not consider the same CVaR levels as in Section 4.2.1; for example, the minimum CVaR attainable in the case of no cardinality constraints is not achievable anymore while imposing cardinality constraints).

We first calculate the minimum CVaR attainable for the specified expected returns and the chosen cardinalities. As it can be seen from Table 4.4, much better CVaR values can be attained in the case of long-short portfolios, even when imposing the same cardinality constraint as in the long-only model. One important aspect is that, in the case of long-short models, although the cardinality constraint is strong, the increase in CVaR (as compared with the case of no cardinality restrictions) is not substantial. In fact, the CVaR range of long-short portfolios with cardinality constraints (either on long only or on long-short positions together) is similar to the CVaR range of the long-short portfolios with no cardinality constraints (see Fig. 4.4 and Table 4.4).

Table 4.4 presents the minimum CVaR levels attainable at the specified expected returns levels, for the three models considered, with the same cardinality constraints. It may be seen that the CVaR of the long-short portfolios is significantly lower (between 3% and 5%) than the CVaR of the long-only portfolios with the same cardinality constraint. In Table 4.4 there is also displayed the composition of the long-short portfolios (the number of long and of short positions). It may be seen that, just by selling 1 or 2 stocks short, a substantial improvement in CVaR can be achieved.

Expected return	Cardinality value	Minimum CVaR (long only)	Minimum CVaR (LS1-cardinality restriction on long positions only)	Minimum CVaR No of short positions in LS1	Minimum CVaR (LS2-cardinality restrictions on long-short positions)	Minimum CVaR No. of long positions in LS2	Minimum CVaR No. of short positions in LS2
0.014034	15	0.070564	0.040221	2	0.040389	13	2
	10	0.070561	0.041149	3	0.043660	8	2
0.018801	10	0.091857	0.049072	2	0.049088	9	1
	8	0.091857	0.049700	2	0.053813	7	1
0.023567	6	0.122467	0.072532	1	0.077272	5	1
	5	0.122512	0.074121	2	0.083122	4	1

Table 4.4. The minimum CVaR levels attainable for long only and long-short models, while holding the same cardinality constraints

For each level of expected return d_1 , d_2 , d_3 , we consider 3 CVaR levels, attainable for all the considered cardinalities and solve the corresponding MVC models.

We notice that the variance that increases at a somewhat greater rate when imposing cardinality restrictions (as compared to the increase in CVaR, which is small). However, the variance of the long-short portfolios is still lower than the variance of their long-only counterparts with the same cardinality constraint (see Fig. 4.4, Tables 4.5, 4.6 and 4.7).

We also calculated the Sharpe and the Sortino ratios of the considered portfolios; these values, together with the corresponding CVaR and variance values are displayed in Table 4.5 (for long only portfolios), Table 4.6 (for long-short portfolios with cardinality constraint on the long positions) and Table 4.7 (for long-short portfolios with cardinality constraint on the total number of positions). Tables 4.6 and 4.7 also display the composition of the long-short portfolios. It can be seen that the Sharpe and the Sortino ratios of the long-short portfolios are substantially higher.

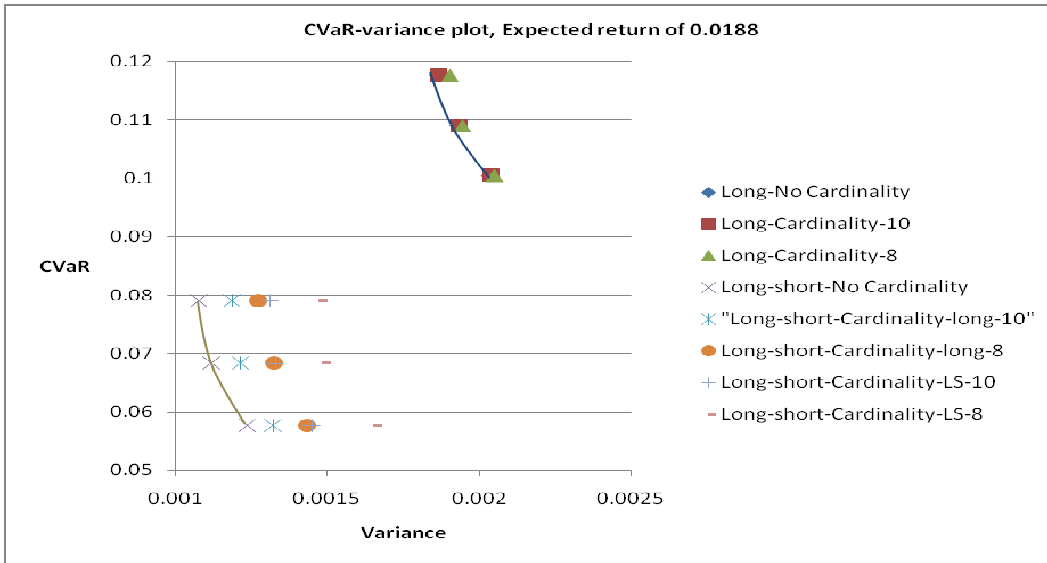


Fig. 4.4: CVaR-variance plot for efficient portfolios with various cardinalities, expected return level = 0.0188

Expected return	Cardinality value	CVaR	Variance	Sharpe ratio	Sortino ratio
0.014034	15	0.0791	0.00126	0.2894	0.6874
	10		0.00129	0.2866	0.6813
	15	0.0876	0.00119	0.2982	0.6959
	10		0.00123	0.2935	0.6896
	15	0.0961	0.00116	0.3016	0.6937
	10		0.00121	0.2956	0.6890
0.018801	10	0.1005	0.00204	0.3334	0.7897
	8		0.00205	0.3324	0.7912
	10	0.1091	0.00193	0.3424	0.8055
	8		0.00194	0.3415	0.8095
	10	0.1177	0.00187	0.3485	0.8080
	8		0.00190	0.3450	0.8135
0.023567	6	0.1314	0.00374	0.3239	0.7222
	5		0.00384	0.3200	0.7076
	6	0.1404	0.00356	0.3322	0.7396
	5		0.00363	0.3290	0.7407
	6	0.1493	0.00343	0.3385	0.7488
	5		0.00351	0.3347	0.7337

Table 4.5. Long only portfolios with cardinality constraints: CVaR, variance and performance measures.

Expected return	Cardinality value	CVaR	Variance	No of short positions	Sharpe ratio	Sortino ratio
0.014034	15	0.049781	0.000914	7	0.3402	0.9367
	10		0.001003	8	0.3247	0.8731
	15	0.05934	0.000828	5	0.3574	0.9276
	10		0.000918	6	0.3394	0.8948
	15	0.0689	0.000796	6	0.3645	0.9186
	10		0.000864	6	0.3499	0.8949
0.018801	10	0.059813	0.001376	5	0.4057	1.0842
	8		0.001492	3	0.3897	1.0365
	10	0.0706	0.001264	6	0.4233	1.0788
	8		0.001383	4	0.4047	1.0562
	10	0.081387	0.001232	5	0.4288	1.0708
	8		0.001335	5	0.4119	1.0225
0.023567	6	0.082113	0.002432	4	0.4018	1.0223
	5		0.002736	4	0.3789	1.0663
	6	0.093293	0.002393	4	0.4051	1.0248
	5		0.00258	3	0.3901	0.9875
	6	0.104473	0.002299	3	0.4133	1.039
	5		0.00253	4	0.394	1.025

Table 4.6. Long-short portfolios with cardinality constraints on the long positions only: CVaR, variance and performance measures.

This was expected, in the case of the Sharpe ratio, since the variance of the long-short portfolios (even with cardinality constraint) is smaller than the variance of the long-only portfolios (with or without cardinality). The smaller Sortino ratio obtained in the case of long short models shows that, even when imposing strong cardinality restrictions, the downside risk of the long-short portfolios is much smaller than that of their long-only counterparts.

To summarize, even when imposing on the long-short portfolios the same cardinality as in the long only model, the long-short strategy still results in better choices, with much lower risk exposures (as measured by variance and especially by CVaR) and higher performance indices (Sharpe and Sortino).

Expected return	Cardinality value	In-sample CVaR	Variance	No of short positions	No of long positions	Sharpe ratio	Sortino ratio
0.014034	15	0.049781	0.001004	3	12	0.3246	0.87519
	10		0.001209	2	8	0.2958	0.782609
	15	0.05934	0.00092	3	12	0.3387	0.89776
	10		0.001079	2	8	0.3131	0.819738
	15	0.0689	0.000854	3	12	0.3519	0.91688
	10		0.001008	2	8	0.3239	0.845193
0.018801	10	0.059813	0.001506	2	8	0.3878	0.995023
	8		0.0017	1	7	0.365	1.008604
	10	0.0706	0.001389	2	8	0.4038	1.060514
	8		0.001555	2	6	0.3817	1.01148
	10	0.081387	0.001359	2	8	0.4083	0.998653
	8		0.001542	2	6	0.3833	1.007955
0.023567	6	0.082113	0.002812	1	5	0.3737	1.029604
	5		Infeasible	N/A	N/A	N/A	N/A
	6	0.093293	0.002756	1	5	0.3775	1.030147
	5		0.0032	1	4	0.3503	0.932923
	6	0.104473	0.002647	1	5	0.3852	0.958898
	5		0.002878	1	4	0.3694	0.867987

Table 4.7. Long-short portfolios with cardinality constraints on the sum of long-short positions: CVaR, variance and performance measures.

4.3.3 Out-of-sample analysis

We analyze the performance of the long and long-short portfolio decisions with cardinality restrictions (described in Section 4.3.2) for the next 18 time periods (Jan 2004 to June 2005) following the date of portfolio selection. We compute the statistics of the out-of-sample returns of these portfolios; they are displayed in Tables 4.8 and 4.9, for the portfolios constructed at the expected return level of 0.0188. Even in the presence of cardinality constraints, long-short portfolio decisions outperform long only portfolio decisions (higher mean, higher minimum, lower standard deviation of out-of-sample returns). Figure 4.5 shows the out-of-sample cumulative returns. It is easily seen that the out-of-sample performance of the long-short portfolios is better.

In-sample expected return	In-sample CVaR	Portfolio details	Cardinality value	Mean	Median	Std dev	Min	Max
0.0188	0.1005	Long only without cardinality restriction	No cardinality restriction	0.0125	0.0138	0.0319	-0.0326	0.0875
		Long only with cardinality restriction	10	0.0125	0.0126	0.0320	-0.0326	0.0872
			8	0.0124	0.0122	0.0325	-0.0346	0.0892

Table 4.8: The long only case. Statistics of the out-of-sample returns of portfolios with in-sample expected return of 0.0188 and CVaR level of 0.1005, various cardinalities.

Expected return	In-sample CVaR	Portfolio details	Cardinality value	Mean	Median	Std dev	Min	Max
0.0188	0.0598	Long-short without cardinality	No cardinality restriction	0.0149	0.0169	0.0272	-0.0269	0.0683
		Long-short with cardinality restriction on long positions	10	0.0144	0.0199	0.0299	-0.0334	0.0764
			8	0.0144	0.0175	0.0299	-0.0321	0.0826
		Long-short with cardinality restriction on long-short positions	10	0.0169	0.0224	0.0298	-0.0315	0.0738
			8	0.0158	0.0192	0.0263	-0.0380	0.0656

Table 4.9: The long-short case. Statistics of the out-of-sample returns of portfolios with in-sample expected return of 0.0188 and CVaR level of 0.0598, various cardinalities.

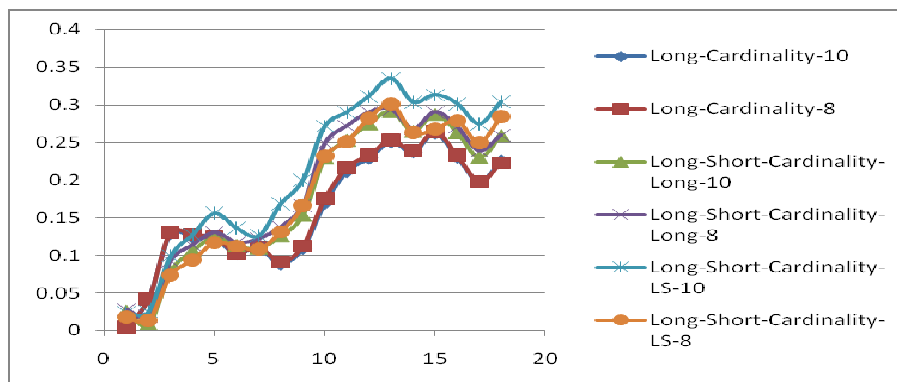


Fig. 4.5: Out-of-sample cumulative returns for 01-01-2004 to 06-01-2005 (portfolios with in-sample expected return of 0.0188 and various cardinality constraints)

5 Conclusions and future directions

In this paper, we have considered a long-short portfolio selection model based on three statistics of the return distribution (mean, variance, CVaR) and in the presence of discrete asset choice constraints of (i) buy and sell thresholds (ii) cardinality restriction on the number of stocks to be held in the portfolio. The proposed decision model has three objectives: the expected return is maximised, while the variance and CVaR are minimised. We chose variance and CVaR mainly because they are well established risk measures that quantify risk from different perspectives: variance measures the deviation around the expected value while CVaR measures the average loss over a specified number of worst cases. The benefits of using both these risk measures, instead of just one of them, have been previously underlined in Roman et al. (2007). Relaxing the no short selling constraint provides the investor with increased choices for portfolio decisions and hence an improvement in the quality of portfolio decisions. In practice, there are many constraints associated with short selling. Such constraints considered in our model include Regulation T margin requirements dictated by regulators, budget restrictions and investor's strategy decisions. For experimental results we further considered full market exposure and 120:20 long-short trading strategy. Threshold on individual positions (long or short) helps in eliminating unrealistic small trades that may otherwise be included in the optimised portfolio. Cardinality restrictions specify user preference to hold a particular number of stocks in the portfolio.

Computationally, the problem reduces to solving a single objective problem in which variance is minimized, while constraints are imposed on the expected value and CVaR. In the practice of portfolio selection, the random variables under consideration are usually represented as discrete and described by realizations under various scenarios. Under scenario models, a CVaR constraint is represented by linear inequality constraints. The constraints associated with short selling and discrete asset choice constraints of threshold and cardinality can be written as linear constraints. Thus, the proposed long-short model with discrete asset choice constraints is formulated as a Quadratic Mixed Integer Programming (QMIP) problem.

Our numerical experiments confirm the advantage of short selling in portfolio construction. Although it is believed that long-short portfolios are riskier than the long only portfolios (due to potentially unlimited losses on the short positions), our study shows that, in practice, by imposing suitable constraints, we can obtain long-short portfolios with much better risk exposures (as described by variance and CVaR) than their long only counterparts. In particular, the CVaR and the downside risk can be substantially reduced by allowing short-selling. The long-short portfolios have a much better performance

than their long-only counterparts, as measured by well-known measures (the Sharpe and the Sortino ratios).

However, the long-short portfolios comprise many more stocks than the long only ones. By introducing cardinality constraints we show that the better performance of long-short portfolios is not only due to the “diversification effect”. Even imposing the same cardinality constraints as for the long only portfolios, the long short portfolios have better return distributions, with much lower risk exposures and higher values of performance measures (Sharpe and Sortino ratios). An important remark is that, in the case of long-short models, by imposing (strong) cardinality restrictions, the corresponding increase in CVaR is very small, as compared to the case of no cardinality restrictions. Our experiments show also considerably better out of sample performance of long-short portfolios.

In this study we have used the covariance matrix determined from historical data. This direct approach for computing the dense covariance matrix based on historical data has been criticized for accuracy and its use in practice has been long debated in the literature. We plan to use a factor model for computing the covariance matrix. The factor model is comparatively more accurate and explanatory; in addition, it is faster, from a computational point of view. Further we plan to make the portfolio selection model even more realistic by introducing index tracking and portfolio rebalancing features. A similar and enhanced version of the long-short model has been implemented for our research sponsor, UBS equity research.

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APPENDIX A

The long-short MVC portfolio selection model under enhanced active equity 120:20 strategy

$$\min \sum_{j=1}^n \sum_{k=1}^n (\sigma_{jk} x_{Lj} x_{Lk} + \sigma_{jk} x_{Sj} x_{Sk} - 2\sigma_{jk} x_{Lj} x_{Sk})$$

Subject to:

$$\sum_{j=1}^n (\mu_j x_{Lj} - \mu_j x_{Sj} + r_c x_{Sj} h_j) \geq d; \quad (d \text{ is the desired expected return level})$$

$$y_i \geq v - \sum_{j=1}^n (r_{ij} x_{Lj} - r_{ij} x_{Sj} + r_c x_{Sj} h_j) \quad \forall i \in 1 \dots T$$

$$\frac{1}{\alpha} \sum_{i=1}^T p_i y_i - v \leq z; \quad (z \text{ is the desired CVaR level})$$

$$(s - \tau) \leq \sum_{j=1}^n (x_{Lj} - x_{Sj}) \leq (s + \tau)$$

$$\sum_{j=1}^n x_{Lj} = 1.2$$

$$\sum_{j=1}^n x_{Sj} = 0.2$$

$$\delta_{Lj} * L_{lower} \leq x_{Lj} \leq \delta_{Lj} * L_{upper}; \quad \forall j \in 1 \dots n$$

$$\delta_{Sj} * S_{lower} \leq x_{Sj} \leq \delta_{Sj} * S_{upper}; \quad \forall j \in 1 \dots n$$

$$\sum_{j=1}^n (\delta_{Lj} + \delta_{Sj}) = k; \quad (k \text{ is the cardinality value})$$

$$\sum_{j=1}^n x_{Lj} + x_c - x_b \leq 1;$$

$$\delta_{Lj} + \delta_{Sj} \leq 1; \quad \forall j \in 1 \dots n$$

$$x_{Lj} \geq 0, x_{Sj} \geq 0 \quad \forall j \in 1 \dots n$$

$$x_c \geq 0, x_b \geq 0$$

$$y_i \geq 0 \quad \forall i \in 1 \dots T$$

$$\delta_{Lj}, \delta_{Sj} \text{ binary}$$

APPENDIX B

The long-short MVC portfolio selection model under margin trading and Regulation T type of constraints

In addition to the variables and parameters already defined in Tables 3.1 – 3.4 we define the following parameters:

p represents percentage margin deposits for long positions.

q represents percentage margin deposits for long positions.

r_b represents risk free rate of borrowing.

r_l represents risk free rate of lending.

$$\min \sum_{j=1}^n \sum_{k=1}^n (\sigma_{jk} x_{Lj} x_{Lk} + \sigma_{jk} x_{Sj} x_{Sk} - 2\sigma_{jk} x_{Lj} x_{Sk})$$

subject to

$$\sum_{j=1}^n (\mu_j x_{Lj} - (1-p)r_b x_{Lj} - \mu_j x_{Sj} + (q+h_j)r_l x_{Sj}) \geq d; \quad d \text{ is the desired expected return level}$$

$$y_i \geq v - \sum_{j=1}^n (r_{ij} x_{Lj} - (1-p)r_b x_{Lj} - r_{ij} x_{Sj} + (q+h_j)r_l x_{Sj}) \quad \forall i \in 1 \dots T$$

$$\frac{1}{\alpha} \sum_{i=1}^T p_i y_i - v \leq z; \quad z \text{ is the desired CVaR level}$$

$$\sum_{j=1}^n (px_{L_j} + qx_{S_j}) \leq 1; \text{ Regulation T requires, } p^* = 0.5, q = 0.5.$$

$$(s - \tau) \leq \sum_{j=1}^n (x_{L_j} - x_{S_j}) \leq (s + \tau)$$

$$\delta_{L_j} * L_{lower} \leq x_{L_j} \leq \delta_{L_j} * L_{upper}; \forall j \in 1 \dots n$$

$$\delta_{S_j} * S_{lower} \leq x_{S_j} \leq \delta_{S_j} * S_{upper}; \forall j \in 1 \dots n$$

$$\sum_{j=1}^n (\delta_{L_j} + \delta_{S_j}) = k; k \text{ is the cardinality value}$$

$$\delta_{L_j} + \delta_{S_j} \leq 1; \quad \forall j \in 1 \dots n$$

$$x_{L_j} \geq 0, x_{S_j} \geq 0 \quad \forall j \in 1 \dots n$$

$$y_i \geq 0 \quad \forall i \in 1 \dots T$$

$$\delta_{L_j}, \delta_{S_j} \text{ binary}$$

*Depending upon our investment strategy we may choose a higher value of p to fully utilize the available fund.

ⁱ Generally, for a multi-objective problem:

$$\text{Max } \{f(x) = (f_1(x), \dots, f_T(x)) : x \in X\},$$

A feasible solution $x^1 \in X$ *Pareto dominates* another feasible solution $x^2 \in X$ if $f_i(x^1) \geq f_i(x^2)$ for all $i \in \{1 \dots T\}$ with at least one strict inequality. x_0 is a Pareto efficient (non-dominated) solution of the multi objective problem if and only if there does not exist a feasible x such that x Pareto dominates x_0 . In other words, a Pareto efficient solution is a feasible solution such that, in order to improve upon one objective function, at least one other objective function must assume a worse value.

ⁱⁱ This is not necessarily the same as “the expected value of losses exceeding VaR at confidence level α ”, as it is defined in earlier papers on CVaR. The two definitions lead to the same results when the distribution of the random variable under consideration is continuous, but differences may appear when the considered distribution has discontinuities –see Acerbi et al. (2002), Rockafellar et al. (2002) for more details.

ⁱⁱⁱ The covariance matrix is always positive definite, provided there is no risk free asset in the asset universe and the assets are not duplicated. Thus the minimization of portfolio's variance over a convex set has a unique optimal solution.

^{iv} d_{\max} is also equal to the highest expected return of the component assets in the portfolio selection problem.