



## TECHNICAL REPORT

CTR/71/07

September 2007

Pricing and evaluating a bond portfolio using a regime switching Markov model

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# Pricing and evaluating a bond portfolio using a regime switching Markov model

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September 18, 2007

## Abstract

We apply a regime switching Markov chain model to determine bond prices. Our work builds upon the work of Thomas, Allen & Morkel-Kingsbury (2002) and Jarrow, Lando & Turnbull (1997). The interest rate process and credit rating migration process are considered. Our aim is to study the price evolution of a portfolio of defaultable bonds. We are interested in determining the density process to compute the Value at Risk (VaR) and Conditional Value at Risk (CVaR) a year ahead. Whereas Thomas et al. describe a model under the pricing measure only, our model takes into consideration both the physical and pricing measures. We also describe the whole (risk free) forward rate curve, with appropriate conditions for the absence of arbitrage, while Thomas et al. describe the spot rate process. Most bond models make use of zero coupon bond prices. By using bond stripping we are able to discover zero coupon bond prices for bonds of different ratings. We use a mathematical programming approach to strip coupons and minimise squared error, subject to no arbitrage constraints.

**Keywords:** Credit risk, Markov model, regime switching model, risk measures, Value at Risk (VaR), Conditional Value at Risk (CVaR).

**JEL Classification** G12, G13.

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# Contents

<b>1</b>	<b>Introduction and background</b>	<b>3</b>
1.1	Structural models . . . . .	3
1.2	Reduced form models . . . . .	4
1.3	The Jarrow Lando & Turnbull model and other related models . . . . .	5
1.4	Motivation and outline of the paper . . . . .	6
<b>2</b>	<b>Formulation of the model and its components</b>	<b>7</b>
2.1	The states of the economy . . . . .	7
2.2	The interest rate process . . . . .	8
2.3	The credit migration process . . . . .	13
2.4	Bond price formulae . . . . .	13
<b>3</b>	<b>Analysis of historical data and calibration of models</b>	<b>16</b>
3.1	The states of the economy . . . . .	16
3.2	The interest rate process . . . . .	17
3.3	Computing zero coupon bond prices . . . . .	20
3.4	The credit migration process . . . . .	21
<b>4</b>	<b>Evaluation of the bond portfolio</b>	<b>22</b>
4.1	Pricing and yield curve simulation . . . . .	23
4.2	Results . . . . .	24
<b>5</b>	<b>Concluding remarks</b>	<b>25</b>
<b>6</b>	<b>Acknowledgements</b>	<b>25</b>
<b>A</b>	<b>Stripping coupons using quadratic programming QP</b>	<b>28</b>
<b>B</b>	<b>Credit migration process calibration</b>	<b>32</b>
B.1	QP initial credit fit . . . . .	32
B.2	QP recursive credit fit . . . . .	33
<b>C</b>	<b>Results of calibration of credit migration process</b>	<b>34</b>
C.1	Physical world measure . . . . .	34
C.2	Pricing measure . . . . .	38

# 1 Introduction and background

Credit risks which refer to the potential for counterparties to default on their financial obligations, form a significant part of the risk exposure for investors, banks and other financial institutions. The measurement and management of these risks are important considerations. The early 1990s saw the growth of credit derivatives which are structured to isolate and transfer credit risk between parties providing an effective tool for risk management since they allow institutions to change their risk profiles. Credit derivatives markets are the fastest growing derivatives markets and more sophisticated, complex products are constantly being developed. It is not always immediately clear how the repackaged risks are dispersed, as we have witnessed recently in the financial markets. Further the Basle accord requires banks to hold regulatory capital to meet their credit risk exposures. The methods specified to calculate these, are simple, broad methods which tend to overstate the required capital and do not model the underlying credit processes. Banks have the freedom to develop their own quantification methods if they can be justified to regulators and of course it is also in the banks interests' to hold suitable levels of capital for their credit risks. There is a need for appropriate models to price credit sensitive assets and quantify the risks of credit sensitive portfolios.

As Rogers (1999) notes "The first and most important thing to realise about modelling of credit risk is that we may be trying to answer questions of two different types."; namely (a) management of credit risks, (b) pricing of credit risks. Risk management questions draw on the probability of credit events and subsequent expected losses under the statistical physical world measures. In contrast, to derive arbitrage-free prices of credit risky assets we rely on the development of artificial "risk neutral" or martingale measures. Arbitrage free martingale pricing methods for contingent claims were developed by Harrison & Krepps (1979) and Harrison & Pliska (1981) and these generalise the risk neutral replicating portfolio arguments first developed by Black-Scholes and Merton placing them in a broader framework.

In this paper we consider both the pricing and management of a portfolio of defaultable bonds. These are subject to credit, interest rate, market, liquidity and other risks. Effective models should give an appropriate integration of significant risks, so most deal with interest rate and credit risks, these being the most relevant with many drawing on the existing well developed literature for default-free term structure models. We also seek to integrate market risks to some extent by considering the state of the economy.

Credit risk models answer two main questions. (i) How should we model random credit events? Two of particular interest are default and downgrading of assets. (ii) What is the payout process given a credit event? Often this is the recovery process on default. Two main forms of credit risk models have emerged. We outline these models in sections 1.1 and 1.2. In section 1.3 we outline the Jarrow, Lando and Turnbull model (1997) and discuss other related models.

## 1.1 Structural models

The structural approach has evolved from the classic work of Merton (1974) and is based on the idea that a company's debt forms a contingent claim on the company's assets so is often also called the "firm value" approach. In Merton's model a firm is funded by equity and a single issue of a zero coupon bond which matures at a fixed future date. Default can only occur at maturity and is triggered if the firm's assets are insufficient to repay the face value of the debt. The debt can be valued as the present value of the face value of the debt minus the value of a European put option on the firm's asset value, which is modelled as a geometric Brownian motion. Thus it is also sometimes referred to as the "option-theoretic approach". The arbitrage-free price is obtained using a similiar argument to that used to derive the Black-Scholes-Merton prices for European put and call options and the formulae are naturally very similiar.

Merton's model, however, ignores interest rate risk. Shimko, Tejima, & van Deventer (1993) develop a model with a Vasicek (1977) model for the short interest rate whereas Wang (1999) uses the Cox-Ingersoll-Ross (CIR) (1985) model. Szatzschneider (2000) also uses the CIR process but unlike Wang assumes dependence between the default process and interest rate process. In reality we also expect that some bonds will pay

coupons. Geske (1977, 1979) uses compound options to develop such a framework. Further we would expect firms to issue more than one type of debt. Vasicek (1984) considers both short and long term debt.

Another class of structural models known as “first passage time” models, specify a barrier process and default is defined to occur when the value of the firm falls below this barrier process. The process may be stochastic or deterministic and may be specified endogenously or exogenously to the firm value process. The advantage is that it allows bankruptcy before maturity of the debt and more flexibility in specifying the recovery payoff. So we can represent more accurately actual debt covenants and other factors such as bankruptcy costs and taxes. Black & Cox (1976), for example, use a deterministic exogenously given boundary and so allow for a safety covenant that forces the firm to declare bankruptcy or reorganise, if assets fall below a given level. They also consider two classes of debt, senior high ranking debt and subordinated junior debt and examine the situation when assets can be sold to meet debt payments. These models naturally give a basis for the study of optimal capital structures for firms. Leland(1994) and Leland & Toft(1996) develop models with allowances for tax benefits of debt funding and bankruptcy costs. Sometimes creditors may renegotiate the debt structure rather than force bankruptcy; see Mella-Barral & Peraudin (1997) Anderson & Sundaresan (1996). Cathcart & El-Jahel (1997) replace the firm value process with a signaling process, which can be chosen to describe relevant economic variables for each issuer and this makes it suitable for valuing issues which do not have easily observable assets. They also use a CIR process for the short rate. Longstaff & Schwartz (1995), and Kim, Ramaswamy & Sundarsen (1993) also incorporate stochastic short rates. The added complexity of “first passage time” models means that we normally employ probabilistic techniques as normally closed form solutions may not be found.

Structural models are appealing as they yield natural economic interpretations. The main criticisms are, the assumption that the firm value process can be observed is questionable and the use of option theoretic pricing relies on the belief that the firm’s value is a tradeable asset. Also they are firm specific models and pricing of one type of debt requires consideration of that firm’s capital structure, debt covenants, tax position and bankruptcy costs. So they can be impractical for pricing portfolios of debt. Since the asset value process is continuous, default is predictable and credit spreads fall as the bond reaches its maturity date. Nevertheless, empirical evidence does not support this; see for example, Jones, Mason & Rosenfeld (1984). Madan & Unal (1998) try to overcome this by introducing jumps in the firm value process as do Zhou (1996) & Schönbucher (1996). Jarrow et al. (1997) note that the models are not suitable for pricing derivatives where payoffs are dependent on the credit rating of debt issues. However, structural models naturally incorporate market risk as credit events are dependent on movements of the firm value process.

## 1.2 Reduced form models

Reduced-form models unlike structural models do not consider the value of the firm or its capital structure, so overcome the problem of needing to observe the firm’s value process. Credit events are defined to occur in terms of exogenously specified jump processes and recovery rates are also given exogenously. They can be split between intensity-based models and credit migration models.

Intensity based models consider only the default event via hazard rates which are normally Cox point processes, where the random time of the first jump of the process represents the time of default. So default occurs completely unexpectedly resolving the problem of default being predictable in structural models. Jarrow & Turnbull (1995), Litterman & Iben(1991) and Duffie & Singleton (1999) are examples of such models. The intensity of default can be a stochastic function dependent on other relevant processes such as the interest rate, company’s share price, and the credit ratings of the debt.

In an ideal world we would have access to all relevant information that affects a bond issuer’s probability of defaulting. But in reality investors do not have access to this information and even if they did incorporating all the information may lead to unduly complex models. However, commercial credit agencies have access to firm-specific information such as current and projected sales, capital structure and ability of management. These items of information are used to categorise debt within credit rating classes and these give us some

indication of the debt's quality. Credit migration models replicate the movement of debt between homogeneous groups. Normally these are taken to be the credit ratings. Jarrow, Lando & Turnbull (1997) were the first to introduce such a model using a Markov chain where the ratings represent the state space of the process with a further state representing bankruptcy. The random default time is then characterised by the first passage time of this state. The introduction of the migration process naturally introduces stochastically varying default intensities. So we could regard them as a special category of intensity models. Jarrow et al. (1997) observe that models that do not use the migration process may understate the probability of default as they ignore the probabilities of changing rating class and then defaulting. They are also suitable when we wish to price derivatives where the payoff is dependent on the underlying's credit rating.

On default firms are assumed to be either liquidated or restructured, so the bondholder receives an unknown random value. The recovery process is defined exogenously and can be deterministic or stochastic. Common assumptions for defaultable bonds include zero recovery, a fraction of the value of an equivalent default-free bond (recovery of treasury), a fraction of the pre-default value of the defaultable bond (recovery of market value) and a fraction of the face value of the defaulted bond (recovery of face value). Duffie & Singleton (1999) is an example of an intensity approach where they use a stochastic process in modelling the recovery process.

For all reduced form models once we have defined a default process, interest rate process and recovery process, the price of a credit risky asset is derived as the expected value of the payoff under a suitable martingale measure.

A major criticism of reduced form models is that they do not take into account the firm's value or capital structure, which could be relevant. Some models try to incorporate aspects of both the reduced-form and structural approaches. Madan & Unal (2000) consider a model where the intensity process is a function of the equity value process. Duffie & Lando (1998) show that using a structural model with imperfect observations of the firm's asset, default is a Poisson like process.

### 1.3 The Jarrow Lando & Turnbull model and other related models

Our work utilises a group of models which are inter-related and can be traced back to the work of Jarrow, Lando & Turnbull (1997)(JLT model), so we outline briefly the discrete time version of this model. We are concerned with pricing zero coupon defaultable bonds. Consider a filtered probability space  $(\Omega, \mathcal{F}, P)$  with fixed time horizon  $T^*$ . The credit migration process is represented by a time homogenous Markov chain  $C_t$ , with state space  $\{0, 1, \dots, M\}$  where 0 represents the rating for default-free government bonds and  $M$  represents bankruptcy. The transition matrix is  $R = [p_{ij}]$  where  $p_{ij} = P\{C_{t+1} = j | C_t = i\} \forall i, j \in \{0, \dots, M\}$  and the random default time is defined by the first passage time of  $C_t$  into state  $M$ . The model is flexible as any interest rate process can be used to model the short rate, provided it is adapted to a sub-filtration of  $\mathcal{F}$ . In order to introduce arbitrage-free pricing, JLT assume the existence of a unique equivalent martingale measure  $\tilde{P}$  such that all zerocoupon bonds discounted by the short rate follow  $\mathcal{F}$ -martingales under  $\tilde{P}$ . On default they assume fractional recovery of treasury with a constant fraction  $\eta$ .

One major assumption made by JLT is that the processes for the interest rate and credit migrations are independent under the martingale measure. There is difficulty in fitting the model and to make this easier they assume that the credit migration process remains a Markov process under the change of measure from  $P$  to  $\tilde{P}$ , though it does not remain time homogenous. Bielecki & Rutkowski (2002, chapter 12) note this assumption has some consequences for the information structure of the processes. Generally we cannot assume that under a change of measure the Markov property is preserved. The appropriate random Radon-Nikodym derivative may be adapted to the filtration of the whole model, not just to that of the Markov process. That is, it may involve information from all random processes in the model. However, if it is adapted to the same filtration as the Markov process in the physical world then the property is preserved. Suppose the filtration can be decomposed as  $\mathcal{F} = \mathcal{F}^C \otimes \mathcal{F}^r$  where  $\mathcal{F}^C$  is the filtration generated by the credit migration process and  $\mathcal{F}^r$  is the filtration generated by the interest rate process, then naturally the property is preserved. With the assumption of independence of the two processes the filtration can be decomposed as above.

The change to a risk neutral measure for the credit migration process is described by  $\tilde{p}_{ij}(t, t+1) = p_{ij}\pi_i(t)$ ,  $\forall i, j \in \{0, 1, \dots, M\}$  where  $i \neq j$ .  $\pi_i(t)$  is a deterministic function referred to as the risk premia adjustment. The Markov structure is maintained and it is assumed that the change of measure depends only on the current state and is independent of the state the process moves to. This may not be true in practice but is adopted for tractability. Prices are derived by considering cash flows under the measure  $\tilde{P}$ .

The physical world transition matrix  $R$  and the recovery constant  $\eta$  can both be estimated from historical data. Then model calibration involves estimating the parameters  $\pi_i(t) \forall t \in \{0, 1, \dots, T\}$ . This can be done by assuming observed risky bond prices represent arbitrage-free prices and the risk premia adjustment is computed using a recursive procedure. The adjustment can be defined in different ways to that adopted by Jarrow et al. Lando (1999) discusses different ways to derive risk neutral measure given an empirical physical world matrix. Kijima & Komoribayashi(1998) observe that JLT's method can introduce bias to the model because the probability of default is low for highly rated bonds. In fact, Jarrow et al. find they have negative risk premia adjustments, which cannot be allowed if the measures are to be equivalent. Consider the probability of a bond in state  $i$  changing state over  $(t, t+1]$ . As noted above Jarrow et al. describe the change of measure from physical to risk neutral as  $\tilde{p}_{ij}(t, t+1) = p_{ij}\pi_i(t)$  where  $i \neq j$ . In order that the probabilities of all possible events sum to 1 they define  $\tilde{p}_{i,i}(t, t+1) = 1 - \sum_{i \neq j} \tilde{p}_{i,j}(t, t+1)$ . Kijima & Komoribayashi give an alternative description of the change of measure as  $\tilde{p}_{ij}(t, t+1) = p_{ij}\pi_i(t)$  where  $i \neq M$  so that the probability of defaulting becomes the balancing probability,  $\tilde{p}_{i,M}(t, t+1) = 1 - \sum_{i \neq M} \tilde{p}_{i,j}(t, t+1)$ . They find that the numerical results under this assumption are good.

To make the model tractable, the migration process is assumed to be a time homogenous Markov chain under the physical world measure. Conditional Markov models allow dependence of transition rates on other relevant factors, such as business cycles, which vary over time. Wei (2000) adopts such a model. His model assumes that the risk premia are constant over time whilst the typical transition matrices vary over time, as he notes there is no good reason for the premia to vary over time, but it seems more reasonable to assume transition probabilities will.

Most term structure models are described in terms of and built from zero coupon bond prices, but in reality the majority of risky bonds traded and available in the market pay coupons. To fit models the coupons must be "stripped" out and the underlying implied zero coupon bond prices derived. It is important that these prices are then consistent, such that they remove or at least reduce arbitrage opportunities. JLT bucket defaultable bonds into categories according to coupons payable and term to maturity and then use a bootstrapping procedure to derive zero coupon bond prices. However, they observe that the method leads to mispricing and makes it difficult to calibrate their model using this derived data. Thomas, Allen & Morkel-Kingsbury (2002) use the method outlined in Allen, Thomas & Zheng (2000) to strip these coupons; they apply linear programming to derive appropriate yields that minimise the error between observed bond prices and those predicted for coupon bonds.

Thomas et al. (2002) develop the JLT model to incorporate a hidden Markov chain represents the underlying economic conditions and can be regarded as a "regime switching" model. This approach resolves some of the problems of the JLT model. The assumption that the interest rate process is independent of the credit migration process is arbitrary and it is very likely that the two processes are in fact related to each other. For instance, it may be a sensible belief that high interest rates with a recessive economy may result in higher default levels. Duffee (1996) states that "defaults are primarily driven by the business cycle which derives variations in the financial variables on which derivatives are priced." The use of the Markov chain removes the strong assumption of independence of credit migrations and the default-free short rate and also introduces market risk to the model.

## 1.4 Motivation and outline of the paper

We have adopted Thomas et al.'s model (2002) model to price a portfolio of zero coupon bonds, characterised by the proportions of bonds held in each credit rating category. We use the same model under the

statistical measure to simulate the distribution of the portfolio at future dates and calculate the respective Value at Risk (VaR) and Conditional Value at Risk (CVaR). Thus we investigate the evolution of the risk process over time. Further we use quadratic programming to strip coupons from bonds when deriving the current zero coupon bond prices, rather than the linear programming method used by Allen et al. (2000).

We use a regime switching model which provides some allowance for correlations in default by different firms. The underlying assumption is that when the economy is in recession default rates are higher. So on observing a large number of defaults we may conclude the economy is in the “bad” state and use the default rates appropriate for this state. The number of defaults observed influence the probability of future default and we have modelled correlations between defaults to some extent. To keep the model tractable we do not introduce any further structure to model the correlations. For example there could be further correlations for firms in similar industries. It seems reasonable to assume that the correlations are largely determined by the state of the economy. Di Graziano & Rogers (2005) note that the use of a hidden Markov chain (HMM) can avoid the need to use copula modelling to introduce dependence between assets. Giampieri, Davis & Crowder (2005) use a HMM to introduce default correlations and use a binomial model to represent the number of defaults at any time. They also extend the model to consider default correlation due to industry specific risks.

The rest of the paper is organised as follows. In section 2 we consider the pricing of bonds using three overlapping models. In section 2.1 we consider a Markov chain model to describe a regime switching economy, in section 2.2 the interest rate model is outlined and in section 2.3 the credit rating migration model is presented. The resulting pricing formula is set out in section 2.4. In section 3 we describe how using historical data the parameters are estimated for individual assets in our given portfolio; some issues of calibrating the model are also discussed. In section 4 we introduce VaR and CVaR as risk measures. The model parameters are used to simulate and project forward in time the evolution of the density function for our credit portfolio. The VaR and CVaR profiles a one period forward is thus computed. In section 6 we discuss the results and conclude. Appendix A outlines the quadratic programming method used to strip coupons to derive zero coupon bond prices. Appendix B outlines the fitting procedure used to calibrate the pricing measure for the credit migration process.

## 2 Formulation of the model and its components

The model we develop is based on Thomas et al. (2002). It is a discrete time model built by combining three separate models. They are concerned with pricing bonds and present the model under the risk neutral measure. We consider it under both the risk neutral and physical measures. In our modelling framework we consider the filtered probability space  $(\Omega, \mathcal{F}, P)$  with fixed time horizon  $T^*$  and time set  $S^T = \{0, \dots, T^*\}$ , where  $P$  denotes the physical measure as well as the probability space  $(\Omega, \mathcal{F}, \tilde{P})$  where  $\tilde{P}$  represents the risk neutral pricing measure.

Any bond can be viewed as a portfolio of zero coupon bonds and so most models seek to model the behaviour of zero coupon bonds, as the model presented does. In reality, however, few zero coupon bonds are actually sold and most bonds available in financial markets are coupon bonds. Calibration then involves stripping coupons from prices of bonds available to derive zero coupon bond prices. We discuss this further in appendix A.

### 2.1 The states of the economy

A two-state time homogenous Markov chain  $C_t^E$  describes the state of the economy. The state space is  $S^E = \{G, B\}$  where the two states  $G, B$  represent good and bad economies. The transition probabilities under the physical measure are defined by

$$(1) \quad P\{C_{t+1}^E = G | C_t^E = G\} \equiv p_E^{GG} \quad P\{C_{t+1}^E = B | C_t^E = G\} \equiv p_E^{GB} = 1 - p_E^{GG} \quad \forall t \in S^T$$

$$(2) \quad P\{C_{t+1}^E = B | C_t^E = B\} \equiv p_E^{BB} \quad P\{C_{t+1}^E = G | C_t^E = B\} \equiv p_E^{BG} = 1 - p_E^{BB} \quad \forall t \in S^T.$$

This Markov chain may be hidden because in practice it is unlikely that we would be able to observe directly whether the economy is in a good or bad state. Instead we might observe another process which is a function of this Markov chain and so it is hidden from us. For example, we may observe stock market indices or productivity indices for a country. We can then discover the optimal estimates for the state of the Markov chain using filtering techniques. For simplicity we assume that the chain is not hidden. Mamon & Rodrigo (2005) is an example of a regime switching model which uses an HMM to model the state of the economy. We define the equivalent probabilities under the risk neutral measure and assume the process remains Markov and time homogenous under this measure.

$$(3) \quad \tilde{P}\{C_{t+1}^E = G | C_t^E = G\} \equiv \tilde{p}_E^{GG} = \pi_E^G p_E^{GG} \quad \tilde{P}\{C_{t+1}^E = B | C_t^E = G\} \equiv \tilde{p}_E^{GB} = 1 - \tilde{p}_E^{GG} \quad \forall t \in S^T$$

$$(4) \quad \tilde{P}\{C_{t+1}^E = B | C_t^E = B\} \equiv \tilde{p}_E^{BB} = \pi_E^B p_E^{BB} \quad \tilde{P}\{C_{t+1}^E = G | C_t^E = B\} \equiv \tilde{p}_E^{BG} = 1 - \tilde{p}_E^{BB} \quad \forall t \in S^T$$

where  $\pi_E^G$  and  $\pi_E^B$  are risk premia adjustments which define the change of measure. For the processes to be equivalent we require  $\pi_E^G > 0$  and  $\pi_E^B > 0$ .

## 2.2 The interest rate process

The interest rate process adopted in this paper is the discrete time process described in Jarrow (2002). We have, however, made some changes to the way time is represented. In our model the time intervals between possible maturity dates differ to those intervals between the times at which bond prices are observed. Jarrow notes that when calibrating the discrete time interest process it is convenient to reparameterise the discrete model in terms of its continuous limit. So we present the discrete model and also describe its continuous limit to aid calibration. We do not, however, use the continuous time process to derive closed form pricing equations.

We initially describe the interest rate process, given the economy is in a particular state and assume that it stays within that state over the period of observation. The process can equivalently be described by the evolution of Treasury zero coupon bond prices, forward rates or spot rates. We introduce an auxiliary non-recombining binomial tree  $C_t^I$  which describes the underlying stochastic structure of these processes. Thus we have a discrete one factor model. The bond price, forward rate and spot rate processes are then given in respect of this auxiliary tree.

We let  $S_t^I$  denote the state space at time  $t$ . At time 0 let  $S_0^I = \{0\}$ . At time 1,  $S_1^I = \{u, d\}$  and at time  $t$ ,  $S_t^I$  is given by considering each  $s_{t-1} \in S_{t-1}^I$ , and its members are given by  $s_{t-1}u$  and  $s_{t-1}d$ . We denote by  $p_I(t, s_t)$  the probability of an upward jump over time  $(t, t+1]$ , under the physical measure, where the current state of the process is  $s_t$ . This means that the process is a Markov chain. Figure 1 shows the evolution of the tree over  $t = 1, 2$  and 3.

A zero coupon bond price is dependent on two timepoints, namely the time at which a price is current,  $t$  and the time that the bond matures,  $T$ . The time  $t$ , at which the price is current is the time set that we have considered so far. Let us consider both time sets separately. The discrete time points at which we observe prices are  $t_0, t_1, \dots, t_m$  and assume that the time intervals or timebuckets are the same and is denoted by  $\delta t$ . It is natural to measure time in years, typically  $\delta t = \frac{1}{250} = 0.004$ . Where the meaning is clear, to simplify the notation we use  $t_0 = 0, t_1 = 1$  etc... The discrete times at which a bond can mature are  $T_0, T_1, \dots, T_k$  where we let  $T_0 = t$  the current time. Define  $\Delta_0$  to be the length of time between  $T_0$  and  $T_1$  and more generally define  $\Delta_i$  to be the time between  $T_i$  and  $T_{i+1}$ . If the set  $\Delta = \{\Delta_0, \Delta_1, \dots, \Delta_{k-1}\}$  is given then  $T_1 = T_0 + \Delta_0$  and  $T_{n+1} = T_n + \Delta_n$ .

The zero coupon bond price process is described at each time point  $t$  and possible state  $s_t$ , by a vector of prices of different maturities. This effectively describes the yield curve at each point in time and state. We let  $Z_t^{T_n}(s_t)$  be the time  $t$  zero coupon bond price for a bond maturing at time  $T_n$ ,  $n \in (1, \dots, k)$  given  $C_t^I$  is in state  $s_t$ . Over a time interval the price changes are dependent on the movement of  $C_t^I$ . We describe the evolution of the prices under the physical measure by

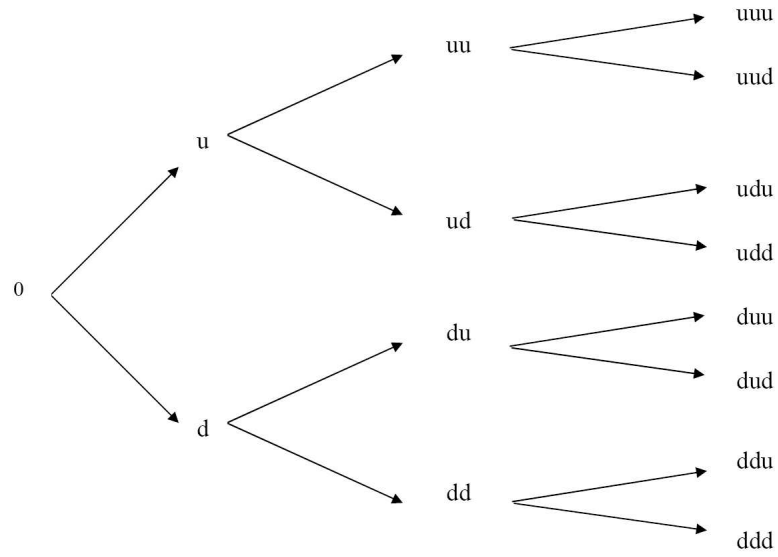


Figure 1: Evolution of state space tree over  $t = 0,1,2,3$

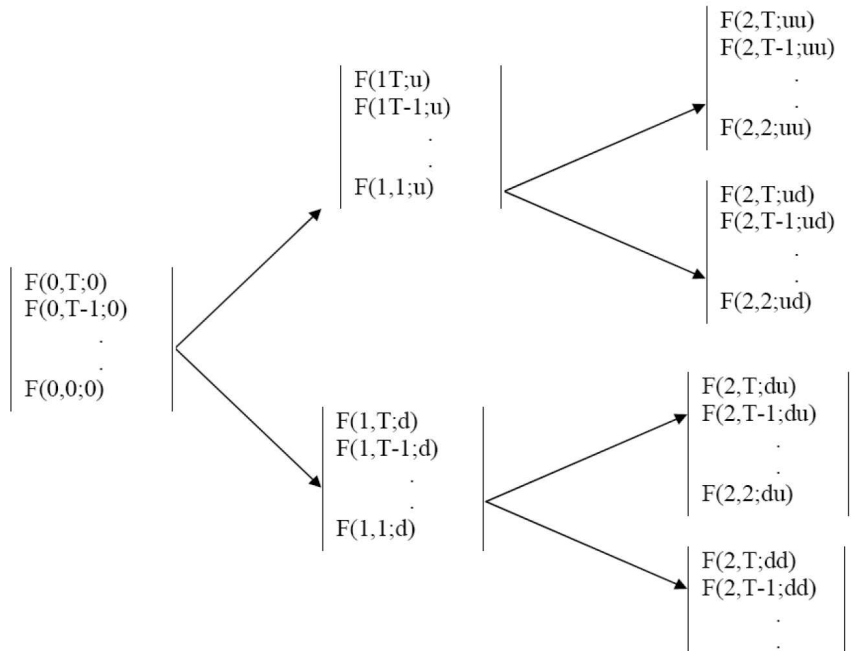


Figure 2: Evolution of forward rates over  $t=1,2$

$$Z_{t+1}^{T_n}(s_{t+1}) = \begin{cases} Z_t^{T_n}(s_t)u(t, T_n; s_t) & \text{if } s_{t+1} = s_t u \text{ with probability } p_I(t, s_t) \\ Z_t^{T_n}(s_t)d(t, T_n; s_t) & \text{if } s_{t+1} = s_t d \text{ with probability } 1 - p_I(t, s_t). \end{cases}$$

The forward rate process is given, by a vector of forward rates for different maturities and describes the entire yield curve. This is illustrated in Figure 2. The time  $t$  annualised forward rate over  $[T_n, T_n + \Delta_n]$  is  $F(t, T_n; s_t)$ , given  $C_t^I = s_t$ . The evolution of the rates is

$$F(t+1, T_n; s_{t+1}) = \begin{cases} F(t, T_n; s_t)\alpha(t, T_n; s_t) & \text{if } s_{t+1} = s_t u \text{ with probability } p_I(t, s_t) \\ F(t, T_n; s_t)\beta(t, T_n; s_t) & \text{if } s_{t+1} = s_t d \text{ with probability } 1 - p_I(t, s_t) \end{cases} \text{ where } n \in \{1, \dots, k-1\}.$$

Define  $r(t; s_t)$  to be the spot rate over the time interval  $(t, t+1]$ . Assume that for the shortest maturity bond  $Z_t^{T_1}(s_t)$  the return over  $(t, t+1]$  is certain and is the spot rate. Define

$$(5) \quad u(t, T_1; s_t) = d(t, T_1; s_t) = \left[ \frac{1}{Z_t^{T_1}(s_t)} \right]^{\frac{\delta t}{\Delta_0}} \text{ and } r(t, s_t) = \left[ \frac{1}{Z_t^{T_1}(s_t)} \right]^{\frac{1}{\Delta_0}}.$$

The relationship between the forward rates changes and the zero coupon bond price changes is

$$(6) \quad \alpha(t, T_n; s_t) = \left[ \frac{u(t, T_n; s_t)}{u(t, T_n + \Delta_n; s_t)} \right]^{\frac{1}{\Delta_n}}$$

$$(7) \quad \beta(t, T_n; s_t) = \left[ \frac{d(t, T_n; s_t)}{d(t, T_n + \Delta_n; s_t)} \right]^{\frac{1}{\Delta_n}}.$$

Conversely,

$$(8) \quad u(t, T_n; s_t) = \frac{r(t; s_t)^{\delta t}}{\prod_{j=1}^{n-1} \alpha(t, T_j; s_t)^{\Delta_j}}$$

$$(9) \quad d(t, T_n; s_t) = \frac{r(t; s_t)^{\delta t}}{\prod_{j=1}^{n-1} \beta(t, T_j; s_t)^{\Delta_j}}.$$

When the state of the economy does not change, the discrete time process can be seen to approximate the continuous time process, given certain assumptions. Let  $F_\Delta(t, T_n)$  denote the  $t$  forward rate over  $[T_n, T_n + \Delta_n]$ . Then  $F_\Delta(t, T_n)^{\frac{1}{\Delta_n}} = e^{f(t, T_n)}$ , where  $f(t, T_n)$  is the annualised continuously compounding forward rate at time  $t$  for maturity  $T$ . So  $f(t, T_n) = \log F_\Delta(t, T_n)^{\frac{1}{\Delta_n}} = \log F(t, T_n; s_t)$ .

In continuous time we are concerned with changes in the continuously compounding forward rates

$$(10) \quad f(t + \delta t, T_n) - f(t, T_n) = \log F_\Delta(t + \delta t, T_n)^{\frac{1}{\Delta_n}} - \log F_\Delta(t, T_n)^{\frac{1}{\Delta_n}}.$$

The continuous process can be described in the real world, by the stochastic differential equation

$$(11) \quad df = \mu(t, T_n)dt + \sigma(t, T_n)dW(t)$$

where,  $\mu(t, T_n)$  is the expected change in continuously compounded forward rates per unit time under the physical measure  $P$  and  $\sigma(t, T_n)$  is the standard deviation of changes in the continuously compounded forward rates per unit time.  $W(t)$  is a standard Brownian motion adapted to time  $t$ . The discrete process in (10) will converge to the continuous process in (11) when

$$(12) \quad E_t \left[ \frac{\log F_\Delta(t + \delta t, T_n)^{\frac{1}{\Delta_n}} - \log F_\Delta(t, T_n)^{\frac{1}{\Delta_n}}}{\delta t} \right] \rightarrow \mu(t, T_n) \text{ as } \delta t \rightarrow 0$$

$$(13) \quad \text{Var}_t \left[ \frac{\log F_\Delta(t + \delta t, T_n)^{\frac{1}{\Delta_n}} - \log F_\Delta(t, T_n)^{\frac{1}{\Delta_n}}}{\delta t} \right] \rightarrow \sigma^2(t, T_n) \quad \text{as } \delta t \rightarrow 0 \quad .$$

The expectations and variances are given under the empirical probabilities. Similarly, under the pricing measure the continuous process is

$$(14) \quad df(t, T_n) = \tilde{\mu}(t, T_n)dt + \sigma(t, T_n)dW(t)$$

where,  $\tilde{\mu}(t, T_n)$  is the expected change in the continuously compounded forward rates per unit time under the pricing measure  $\tilde{P}$  and  $\sigma(t, T_n)$  is the same under both measures because of the assumption that the measures are equivalent. The process in (14) is approximated by the discrete time process when

$$(15) \quad \tilde{\mathbb{E}}_t \left[ \frac{\log F_\Delta(t + \delta t, T_n)^{\frac{1}{\Delta_n}} - \log F_\Delta(t, T_n)^{\frac{1}{\Delta_n}}}{\delta t} \right] \rightarrow \tilde{\mu}(t, T_n) \quad \text{as } \delta t \rightarrow 0$$

$$(16) \quad \widetilde{\text{Var}}_t \left[ \frac{\log F_\Delta(t + \delta t, T_n)^{\frac{1}{\Delta_n}} - \log F_\Delta(t, T_n)^{\frac{1}{\Delta_n}}}{\delta t} \right] \rightarrow \sigma^2(t, T_n) \quad \text{as } \delta t \rightarrow 0 \quad ,$$

where the expectations and variances are given under the risk neutral pricing probabilities.

Define

$$(17) \quad \alpha(t, T_n; s_t) \equiv e^{(\tilde{\mu}(t, T_n; s_t)\delta t - \sigma(t, T_n; s_t)\sqrt{\delta t})}$$

$$(18) \quad \beta(t, T_n; s_t) \equiv e^{((\tilde{\mu}(t, T_n; s_t)\delta t + \sigma(t, T_n; s_t)\sqrt{\delta t})}$$

$$(19) \quad p_I^e(t; s_t) \equiv \frac{1}{2} + \frac{1}{2}\phi(t; s_t)\sqrt{\delta t}$$

where  $\phi(t; s_t)$  is the process for the risk premium. Then the discrete time physical world process has the representation

$$F(t+1, T_n; s_t) = \begin{cases} F(t, T_n; s_t)e^{(\tilde{\mu}(t, T_n; s_t)\delta t - \sigma(t, T_n; s_t)\sqrt{\delta t})} & \text{if } s_{t+1} = s_t u \text{ with probability } \frac{1}{2} + \frac{1}{2}\phi(t)\sqrt{\delta t} \\ F(t, T_n; s_t)e^{((\tilde{\mu}(t, T_n; s_t)\delta t + \sigma(t, T_n; s_t)\sqrt{\delta t})} & \text{if } s_{t+1} = s_t d \text{ with probability } \frac{1}{2} - \frac{1}{2}\phi(t)\sqrt{\delta t}. \end{cases}$$

Then

$$(20) \quad \lim_{\delta t \rightarrow 0} \mathbb{E}_t \left[ \frac{\log F_\Delta(t + \delta t, T_n)^{\frac{1}{\Delta_n}} - \log F_\Delta(t, T_n)^{\frac{1}{\Delta_n}}}{\delta t} \right] = \tilde{\mu}(t, T_n) - \phi(t)\sigma(t, T_n)$$

$$(21) \quad \lim_{\delta t \rightarrow 0} \text{Var}_t \left[ \frac{\log F_\Delta(t + \delta t, T_n)^{\frac{1}{\Delta_n}} - \log F_\Delta(t, T_n)^{\frac{1}{\Delta_n}}}{\delta t} \right] = \sigma^2(t, T_n).$$

As expected, the drift of the physical world process is the risk neutral drift with an adjustment  $\mu(t, T_n) \equiv \tilde{\mu}(t, T_n) - \phi(t)\sigma(t, T_n)$ . The continuously compounding forward rate process is an Itô process adapted to a standard Brownian motion, so under a change of measure only its drift changes. The function  $\phi(t)$  describes this measure change. Both  $\mu(t, T_n)$  and  $\sigma(t, T_n)$  can be estimated from past observations of changes in continuously compounding forward rates.

Under the pricing measure the discrete time process is

$$F(t+1, T_n; s_t) = \begin{cases} F(t, T_n; s_t)e^{(\tilde{\mu}(t, T_n; s_t)\delta t - \sigma(t, T_n; s_t)\sqrt{\delta t})} & \text{if } s_{t+1} = s_t u \text{ with probability } \tilde{p}_I(t; s_t) \\ F(t, T_n; s_t)e^{((\tilde{\mu}(t, T_n; s_t)\delta t + \sigma(t, T_n; s_t)\sqrt{\delta t})} & \text{if } s_{t+1} = s_t d \text{ with probability } 1 - \tilde{p}_I(t; s_t). \end{cases}$$

If we choose  $\tilde{p}_I(t; s_t) = \frac{1}{2}$  at all time points and states then

$$(22) \quad \lim_{\delta t \rightarrow 0} \tilde{\mathbb{E}}_t \left[ \frac{\log F_\Delta(t + \delta t, T_n)^{\frac{1}{\Delta_n}} - \log F_\Delta(t, T_n)^{\frac{1}{\Delta_n}}}{\delta t} \right] = \tilde{\mu}(t, T_n)$$

$$(23) \quad \lim_{\delta t \rightarrow 0} \widetilde{\text{Var}}_t \left[ \frac{\log F_{\Delta}(t + \delta t, T_n)^{\frac{1}{\Delta_n}} - \log F_{\Delta}(t, T_n)^{\frac{1}{\Delta_n}}}{\delta t} \right] = \sigma^2(t, T_n)$$

and the discrete time process will approximate the continuous time process as the size  $\delta t$  of the time steps decreases.

We assume that the market is arbitrage free. This implies the existence of a risk neutral measure, given the market is complete. In the absence of arbitrage we have the following restriction on the zero coupon bond price process

$$(24) \quad \tilde{p}_I(t; s_t) = \frac{r(t; s_t)^{\delta t} - d(t, T_n; s_t)}{u(t, T_n; s_t) - d(t, T_n; s_t)}.$$

The zero coupon bond price parameters can be given in terms of the forward rate parameters by

$$(25) \quad u(t, T_n; s_t) = r(t; s_t)^{\delta t} e^{-\sum_{j=1}^n \tilde{\mu}(t, T_j; s_t) \delta t \Delta_j + \sum_{j=1}^n \sigma(t, T_j; s_t) \sqrt{\delta t} \Delta_j}$$

$$(26) \quad d(t, T; s_t) = r(t; s_t)^{\delta t} e^{-\sum_{j=1}^n \tilde{\mu}(t, T_j; s_t) \delta t \Delta_j - \sum_{j=t+\Delta}^{T-\Delta} \sigma(t, T_j; s_t) \sqrt{\delta t} \Delta_j}.$$

The no arbitrage condition becomes

$$(27) \quad \tilde{p}_I(t; s_t) = \frac{1 - e^{-\sum_{j=1}^{n-1} \tilde{\mu}(t, T_j; s_t) \delta t \Delta_j - \sum_{j=1}^{n-1} \sigma(t, T_j; s_t) \sqrt{\delta t} \Delta_j}}{e^{-\sum_{j=1}^{n-1} \tilde{\mu}(t, T_j; s_t) \delta t \Delta_j + \sum_{j=1}^{n-1} \sigma(t, T_j; s_t) \sqrt{\delta t} \Delta_j} - e^{-\sum_{j=1}^{n-1} \tilde{\mu}(t, T_j; s_t) \delta t \Delta_j - \sum_{j=1}^{n-1} \sigma(t, T_j; s_t) \sqrt{\delta t} \Delta_j}}$$

Given  $\tilde{p}_I(t; s_t) = \frac{1}{2}$  the condition reduces to

$$(28) \quad e^{\sum_{j=1}^n \tilde{\mu}(t, T_j; s_t) \delta t \Delta_j} = \cosh \left( \sum_{j=1}^n \sigma(t, T_j; s_t) \sqrt{\delta t} \Delta_j \right)$$

The proofs of similiar conditions can be found in Jarrow (2002 chapter 4 and chapter 15).

It remains for us to make the interest rate process dependent on the state of the economy. We generalise the structure and assume the processes' parameters are conditional on the state of the economy, so the transition probabilities are described by the following, under the physical measure

$$(29) \quad \begin{aligned} P\{C_{t+1}^I = s_t u | C_t^I = s_t, C_t^E = G\} &\equiv p_I^G(t, s_t) \\ P\{C_{t+1}^I = s_t d | C_t^I = s_t, C_t^E = G\} &= 1 - p_I^G(t, s_t) \end{aligned}$$

$$(30) \quad \begin{aligned} P\{C_{t+1}^I = s_t u | C_t^I = s_t, C_t^E = B\} &\equiv p_I^B(t, s_t) \\ P\{C_{t+1}^I = s_t d | C_t^I = s_t, C_t^E = B\} &= 1 - p_I^B(t, s_t). \end{aligned}$$

The probabilities of jumps under the pricing measure  $\tilde{P}$  have been fixed as  $\frac{1}{2}$  in all cases. Finally, we introduce  $\mu^e(t, T; s_t)$ ,  $\tilde{\mu}^e(t, T; s_t)$ ,  $\sigma^e(t, T; s_t)$  and  $r(t, s_t, e)$  where  $e \in S^E = \{G, B\}$ .

### 2.3 The credit migration process

The third process  $C_t^R$  describes the credit rating migrations of bonds. Suppose there are  $M+1$  credit ratings. The state space is  $S^R = \{0, 1, \dots, M\}$  where 0 represents rating of default-free government bonds and  $M$  represents bankruptcy. The structure is the same as the JLT model. The conditional transition probabilities under the physical measure are time homogenous and depend on the length of the time period. The set of time points to be considered is  $\{t_0, t_1, \dots, t_n\}$ . The transition probabilities are

$$(31) \quad P\{C_{t_{n+1}}^R = j' | C_{t_n}^R = j, C_{t_n}^E = G\} \equiv {}^R p_{jj'}^G(t_{n+1} - t_n) \quad \forall j, j' \in S^R$$

$$(32) \quad P\{C_{t_{n+1}}^R = j' | C_{t_n}^R = j, C_{t_n}^E = B\} \equiv {}^R p_{jj'}^B(t_{n+1} - t_n) \quad \forall j, j' \in S^R$$

$$(33) \quad \sum_{j' \in S^R} {}^R p_{jj'}^G(t_{n+1} - t_n) = \sum_{j' \in S^R} {}^R p_{jj'}^B(t_{n+1} - t_n) = 1 \quad \forall j \in S^R.$$

These probabilities could be defined as being time in-homogenous, but the ratings transition data published by credit rating agencies such as Standard & Poors and Moodys normally assume time homogeneity.

The process under the risk neutral measure is

$$(34) \quad \tilde{P}\{C_{t_{n+1}}^R = j' | C_{t_n}^R = j, C_{t_n}^E = G\} \equiv {}^R \tilde{p}_{jj'}^G(t_n) = \pi_R^G(t_n, j) {}^R p_{jj'}^G(t_{n+1} - t_n) \quad \forall j, j' \in S^R, \quad \text{where } j' \neq M$$

$$\tilde{P}\{C_{t_{n+1}}^R = M | C_{t_n}^R = j, C_{t_n}^E = G\} \equiv {}^R \tilde{p}_{jM}^G(t) = 1 - \pi_R^G(t_n, j) \sum_{j' \neq M} {}^R p_{jj'}^G(t_{n+1} - t_n) \quad \forall j \in S^R,$$

$$(35) \quad \tilde{P}\{C_{t_{n+1}}^R = j' | C_{t_n}^R = j, C_{t_n}^E = B\} \equiv {}^R \tilde{p}_{jj'}^B(t) = \pi_R^B(t_n, j) {}^R p_{jj'}^B(t_{n+1} - t_n) \quad \forall j, j' \in S^R, \quad \text{where } j' \neq M$$

$$\tilde{P}\{C_{t_{n+1}}^R = M | C_{t_n}^R = j, C_{t_n}^E = B\} \equiv {}^R \tilde{p}_{jM}^B(t) = 1 - \pi_R^B(t_n, j) \sum_{j' \neq M} {}^R p_{jj'}^B(t_{n+1} - t_n) \quad \forall j \in S^R,$$

$$(36) \quad \sum_{j' \in S^R} {}^R \tilde{p}_{jj'}^G(t_n) = \sum_{j' \in S^R} {}^R \tilde{p}_{jj'}^B(t_n) = 1 \quad \forall t_n \in \{0, 1, \dots, T\} \quad \forall j, j' \in S^R.$$

We have adjusted the JLT model in line with the analysis in Kijima & Komoribayashi(1998). See section 1.3. So in deriving the change of measure, risk premia are applied to all the one step physical world probabilities except the probability of moving to the default state, where this is defined as the balancing probability.

### 2.4 Bond price formulae

The processes are linked together to form one large Markov chain by considering the process  $(C_t^R, C_t^I, C_t^E)$ . The Markov property holds under both measures given the construction of the component processes. If  $C_t^E$  is assumed to have one state, the model reduces to the JLT (1997) model where the interest process follows the model specified in section 2.2.

Several events can occur in a time period  $(t, t+1]$ . It is assumed that changes in the state of the interest rate process  $C_t^I$  occur first then changes in rating process  $C_t^R$  and changes in the state of the economy  $C_t^E$  last. The assumption is convenient as it allows us to consider the interest rate and ratings processes dependent on the current state of the economy alone, when considering transitions in one time period.

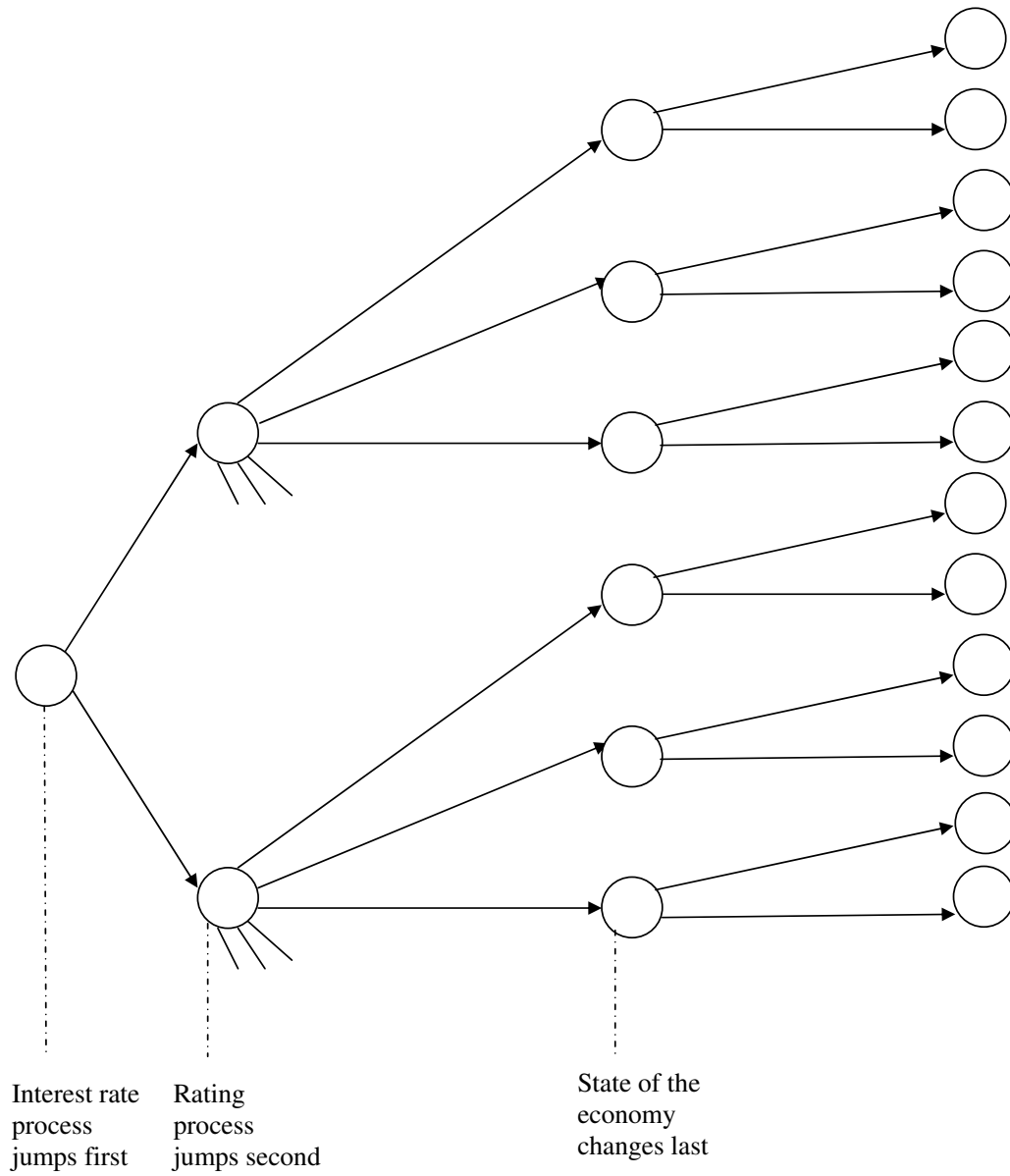


Figure 3: Possible evolution of factors over one time period

On default of a bond it is assumed that a constant fraction  $\eta^e$  of the face value is recovered at the bond's maturity, given the economy is in state  $e \in \{G, B\}$ . We wish to find the arbitrage-free prices for the zero coupon bonds. It is assumed that the discounted values of the prices are martingales under a suitable pricing measure,  $\tilde{P}$ . The price of a bond is the expected, discounted payoff of the bond at maturity, under the pricing measure. We denote by  $Z_t^{T_n}(s_t, j, e)$  the price of a zero coupon bond at time  $t$  which matures at  $T_n$ , given at time  $t$  the interest rate process is at node  $s_t$ , the bond has rating  $j$  and the economy is in state  $e$ . We see that

$$(37) \quad Z_0^{T_n}(0, j, e) = Z_0^{T_n}(0, 0, T_n) A_0^{T_n}(j, e),$$

where  $Z_0^{T_n}(0, 0, T_n)$  is the zero coupon bond price for a Treasury bond with maturity  $T_n$  at time 0. In equation (37)  $A_0^{T_n}(j, e)$  is the expected value at time 0 of the bond's payout at maturity under the pricing measure. It is defined by the recursive formulae

$$(38) \quad A_t^{T_n}(j, e) = \sum_{j' \in S^R} {}^R \tilde{p}_{jj'}^e(t) \left( \sum_{e' \in S^E} \tilde{p}_E^{ee'} A_{t+1}^{T_n}(j', e') \right),$$

which is bounded by,

$$(39) \quad \text{Non-default states;} \quad A_t^{t+1}(j, e) = 1 \quad \text{if } j \neq M$$

and

$$(40) \quad \text{Default states;} \quad A_t^{t+1}(M, e) = \eta^e.$$

The times at which the economy and ratings processes change state do not always coincide. The set of times at which the economy changes is denoted by  $T^E$  and times where the ratings change by  $T^R$ . The set of timepoints we must consider when pricing a bond with the above equations is therefore  $T^E \cup T^R$ .

Thomas et al. model the interest rate process as a spot rate process, without modelling the entire yield curve. Their pricing formulae is

$$(41) \quad Z_t^{T_n}(s_t, j, e) = \frac{1}{1 + r(t, s_t, e)} \left\{ \sum_{s_{t+1} \in S_{t+1}^I} \tilde{p}_I^e(t, s_t, s_{t+1}) \left[ \sum_{j' \in S^R} {}^R \tilde{p}_{jj'}^e(t) \left( \sum_{e' \in S^E} \tilde{p}_E^{ee'} Z_{t+1}^{T_n}(s_{t+1}, j', e') \right) \right] \right\}.$$

The bond price process is bounded by bonds with one time period to maturity,

$$(42) \quad Z_t^{t+1}(s_t, j, e) = \frac{1}{1 + r(t, s_t, e)} \quad \text{if } j \neq M$$

and

$$(43) \quad Z_t^{t+1}(s_t, M, e) = \frac{\eta^e}{1 + r(t, s_t, e)}.$$

Bielecki & Rutkowski (2002 ch. 12) proposed a modification of Thomas et al. model. Consider the vector Markov chain  $(C_t^R, C_t^I, C_t^E)$  and assume it is time homogenous and under the risk neutral framework over time interval  $(t, t + 1]$ . In particular,

$$(44) \quad \tilde{P}\{C_{t+1}^I = s_{t+1}, C_{t+1}^R = j', C_{t+1}^E = e' \mid C_t^I = s_t, C_t^R = j, C_t^E = e\} = \tilde{p}(s_t, s_{t+1}, j, j', e, e').$$

The recursive bond price formulae simplify to

$$(45) \quad Z_t^{T_n}(s_t, j, e) = \frac{1}{1 + r(t + 1, s_t, e)} \sum_{s_{t+1} \in S_{t+1}^I} \sum_{j' \in S^R} \sum_{e' \in S^E} \tilde{p}(s_t, s_{t+1}, j, j', e, e') Z_{t+1}^{T_n}(s_{t+1}, j', e').$$

Bielecki & Rutkowski’s formulation is intuitively appealing as it considers the joint distribution of the enlarged chain. They do not make any assumptions about the statistical relationships between the processes. Our formulation and that due to Thomas et al. rely upon the assumptions set out in this section. Our approach, however, gives a link between the pricing and physical measure which is needed to simulate the future physical world price distribution in calculating VaR and CVaR.

In our model the risk premia adjustments to change from the physical measure to the risk neutral measure are defined entirely by the parameters  $(C_t^R, C_t^I, C_t^E)$  for transitions between  $(t, t + 1]$  and this is why the Markov structure is maintained. Since the interest rate and credit migration processes are dependent on the state of the economy the processes are related and are not independent.

### 3 Analysis of historical data and calibration of models

#### 3.1 The states of the economy

Standard and Poors provided us with historical transition rates between rating classes for the period 1995 - 2005 for US Corporate bonds. This information is used to classify these years as being in one of two states good =  $G$  or bad =  $B$ . Following the approach of Thomas et al. we calculate a “Credit Value” for each year according to the formula

Credit Value for the Year = Percentage downgrades + Percentage defaults - Percentage upgrades.

We rank the years by this value and assign the top half of the years as having state  $B$  and the other half as state  $G$ . That is those with a value above the median (3.45 %) as state  $B$  and the rest state  $G$ . An economic indicator could also be used to classify the economy on a year by year basis. We have assumed that FTSE 100 index annual returns above 10% indicate the economy is in state  $G$ . The results are summarised in Table 1. We may also use other indicators such as Industry Production index, Unemployment data, or Gross Domestic Product data (GDP).

We assume the probability of changing economy is the same under the physical measure and the risk neutral measure. That is  $\pi_E^G = \pi_E^B = 1$  and we could then view the model as an aggregation of two separate sub-models; we switch between these sub-models when the state of the economy changes. The transition probabilities for the economy process are estimated using the observed transition frequencies. From Table 1 we observe that the process starts in state  $G$  five times and four times it moves to state  $G$  again. So, we assign  $p_E^{GG} = \tilde{p}_E^{GG} = \frac{4}{5}$  and then  $p_E^{GB} = \tilde{p}_E^{GB} = \frac{1}{5}$ . Similarly  $p_E^{BB} = \tilde{p}_E^{BB} = \frac{4}{5}$  and  $p_E^{BG} = \tilde{p}_E^{BG} = \frac{1}{5}$ .

Table 1: Classification of the economic states

Year	Credit Value in %	Classification by Credit Value	Classification by FTSE 100 returns
1995	1.80	G	G
1996	-1.73	G	G
1997	0.13	G	G
1998	1.83	G	G
1999	4.88	B	B
2000	7.56	B	B
2001	11.01	B	B
2002	12.87	B	B
2003	8.02	B	B
2004	2.81	G	G
2005	3.45	G	G

### 3.2 The interest rate process

We first calibrate the interest rate process for a period of observation when the economy remains in state  $G$  and then repeat this for a period when it is in state  $B$ . The forward rate process is calibrated and from this the zero coupon bond prices are calculated; Jarrow(2002) notes that this approach gives more stable results. We use observations of changes in the continuously compounding forward rates for different maturities to estimate the volatilities ( $\sigma(t, T_n; s_t)$ ) and physical world drifts ( $\mu(t, T_n; s_t)$ ) at different times and maturities. A simplifying assumption is made that  $\mu(t, T_n; s_t) = \mu(T_n - t)$  and  $\sigma(t, T_n; s_t) = \sigma(T_n - t)$ , that is, these are functions of the time to maturity only. As a result the process is time homogenous and independent of the state, so the binomial tree becomes a recombining tree.

The continuous time physical world process (11) can be approximated by the following discretisation of the small change in the yield curve.

$$(46) \quad \begin{bmatrix} f(t + \delta t, t + \delta t) - f(t, t) \\ f(t + \delta t, T_1) - f(t, T_1) \\ \vdots \\ f(t + \delta t, T_{k-1}) - f(t, T_{k-1}) \end{bmatrix} \approx \begin{bmatrix} \mu(t, t) \\ \mu(t, T_1) \\ \vdots \\ \mu(t, T_{k-1}) \end{bmatrix} \delta t + \begin{bmatrix} \sigma(t, t) \\ \sigma(t, T_1) \\ \vdots \\ \sigma(t, T_{k-1}) \end{bmatrix} \delta W(t)$$

or,

$$(47) \quad \begin{bmatrix} f(t + \delta t, t + \delta t) - f(t, t) \\ f(t + \delta t, T_1) - f(t, T_1) \\ \vdots \\ f(t + \delta t, T_{k-1}) - f(t, T_{k-1}) \end{bmatrix} \approx \begin{bmatrix} \mu(0) \\ \mu(T_1 - t) \\ \vdots \\ \mu(T_{k-1} - t) \end{bmatrix} \delta t + \begin{bmatrix} \sigma(0) \\ \sigma(T_1 - t) \\ \vdots \\ \sigma(T_{k-1} - t) \end{bmatrix} \delta W(t).$$

We define the vector stochastic process

$$(48) \quad \mathbf{x} = \begin{bmatrix} x(0) \\ x(T_1 - t) \\ \vdots \\ x(T_{k-1} - t) \end{bmatrix} \equiv \begin{bmatrix} f(t + \delta t, t + \delta t) - f(t, t) \\ f(t + \delta t, T_1) - f(t, T_1) \\ \vdots \\ f(t + \delta t, T_{k-1}) - f(t, T_{k-1}) \end{bmatrix}.$$

The daily yield curve for a year gives 250 observations, one for each trading day, which are used to derive the values of  $\mathbf{x}$ . We are then able to calculate the sample covariance matrix and sample means for  $\mathbf{x}$ . Here,  $\delta t = \frac{1}{250} = 0.004$ . To calibrate the model the data published on the US Treasury webpage is used, where observations from 2005 give the parameters for the process, given the economy is in state  $G$  and those from 2002 for the state  $B$ . The first three observations for 2005 are shown in Table 2.

Table 2: Three observations of the yield curve in 2005

Date	1 Month	3 Month	6 Month	1 Year	2 Year	3 Year	5 Year	7 Year	10 Year	20 Year
	%	%	%	%	%	%	%	%	%	%
03/01/2005	1.99	2.32	2.63	2.79	3.10	3.28	3.64	3.94	4.23	4.84
04/01/2005	2.05	2.33	2.63	2.82	3.20	3.38	3.72	4.02	4.29	4.91
05/01/2005	2.04	2.33	2.63	2.83	3.22	3.39	3.73	4.02	4.29	4.88

We assume the volatility function is described by the functional form  $\sigma(T - t) = \sigma e^{-\lambda(T-t)}$ . Jarrow(2002) notes that this is equivalent to using a Vasicek model for the short rate. An alternative way of calibrating the volatility function is to carry out a principal components analysis of the observed volatility. We carried this out but several factors would be needed for a reasonable fit. The use of the Vasicek model captures the

expected feature of declining volatility as time to maturity increases.

We have used the procedure described in Jarrow(2002)to calibrate  $\sigma$  and  $\lambda$ . This uses the following identity which holds over the time interval  $(t, t + \delta t]$

$$(49) \quad \text{Var}(\log Z_{t+\delta t}^T - \log Z_t^T - r(t) \delta t) = \frac{\sigma^2(e^{-\lambda(T-t)} - 1)^2 \delta t}{\lambda^2}.$$

From the observations of the yield curve we find values of  $\log Z_{t+\delta t}^T - \log Z_t^T - r(t) \delta t$  and calculate the sample variances,  $v_T$  of these observations at different maturity dates. In the expression for  $v_T$  we introduce an error term  $\epsilon_T$  which we minimise to derive values of  $\sigma$  and  $\lambda$ .

$$(50) \quad v_T = \frac{\sigma^2(e^{-\lambda(T-t)} - 1)^2 \delta t}{\lambda^2} + \epsilon_T \quad \text{for all } T.$$

The calibrated values for  $\sigma$  and  $\lambda$  are shown in Table 3 and the subsequent volatility functions in Table 4.

Table 3: Calibrated values for  $\lambda$  and  $\sigma$

	Economy state G	Economy state B
$\sigma$	7.21830E-03	1.20305E-02
$\lambda$	7.90612E-03	3.24195E-02

Table 4: Calibrated volatility functions

Time to maturity	$\sigma(T_n - t)$ when economy is G	$\sigma(T_n - t)$ when economy is B
0 Month	7.21830E-03	1.20305E-02
1 Month	7.21354E-03	1.19981E-02
3 Months	7.20405E-03	1.19334E-02
6 Months	7.18982E-03	1.18371E-02
1 Year	7.16145E-03	1.16467E-02
2 Years	7.10506E-03	1.12752E-02
3 Years	7.04911E-03	1.09155E-02
5 Years	6.93852E-03	1.02302E-02
7 Years	6.82967E-03	9.58797E-03
10 Years	6.66959E-03	8.69937E-03
20 Years	6.16259E-03	6.29060E-03

For these volatility functions, the risk neutral drift is given by  $\tilde{\mu}(t, T_n; s_t) = \tilde{\mu}(T_n - t)$ , see equation (28) and the results are set out in Table 5. The initial value  $\tilde{\mu}(0)$  is not defined by equation (28), however, we assume it is equal to the observed  $\mu(0)$ .

We can use the values in Tables 4 and 5 to calculate the nodal values of the binomial process, which we use to determine the physical world probabilities of jumps on the tree. It is assumed that the process is time homogenous and independent of the state in the physical world so  $p_I(t; s_t)$  generalises to  $p_I$ , thus

Table 5: Calibrated risk neutral drift functions

Time to maturity	$\tilde{\mu}(T_n - t)$ when economy is G	$\tilde{\mu}(T_n - t)$ when economy is B
0 Month	7.87642E-05	-2.09779E-05
1 Month	4.33627E-06	1.19961E-05
3 Months	1.51484E-05	4.16637E-05
6 Months	3.45163E-05	9.40135E-05
1 Year	7.28958E-05	1.94791E-04
2 Years	1.23004E-04	3.17802E-04
3 Years	1.96767E-04	4.88350E-04
5 Years	2.90733E-04	6.74014E-04
7 Years	4.03525E-04	8.67673E-04
10 Years	6.84802E-04	1.29074E-03

$$(51)\mu^* = \begin{bmatrix} \mu^*(0) \\ \mu^*(T_1 - t) \\ \vdots \\ \mu^*(T_n - t) \end{bmatrix} = \begin{bmatrix} p_I[\tilde{\mu}(0)\delta t - \sigma(0)\sqrt{\delta t}] + (1 - p_I)[\tilde{\mu}(0)\delta t + \sigma(0)\sqrt{\delta t}] \\ p_I[\tilde{\mu}(T_1 - t)\delta t - \sigma(T_1 - t)\sqrt{\delta t}] + (1 - p_I)[\tilde{\mu}(T_1 - t)\delta t + \sigma(T_1 - t)\sqrt{\delta t}] \\ \vdots \\ p_I[\tilde{\mu}(T_n - t)\delta t - \sigma(T_n - t)\sqrt{\delta t}] + (1 - p_I)[\tilde{\mu}(T_n - t)\delta t + \sigma(T_n - t)\sqrt{\delta t}] \end{bmatrix}$$

where  $\mu^*$  is a vector of the sample means of observed  $\mathbf{x}$  (defined by equation 48). The probability  $p_I$  is found as the value that minimises the difference between the observed and calculated physical world drift values. The computed values of  $p_I$  are shown in Table 6.

Table 6: Calibrated  $p_I$ 

	Economy state G	Economy state B
$p_I$	0.468358	0.533945

### 3.3 Computing zero coupon bond prices

Zero coupon bond prices for different rating classes at different maturity dates are computed so that the credit migration process can be calibrated. The maturity dates for the zero coupon bond prices are  $\frac{1}{12}$ ,  $\frac{3}{12}$ ,  $\frac{6}{12}$ , 1, 2, 3, 5, 7, 10 and 20 years. The granularity of the dates are finer at the start of the set. This structure seeks to model the immediate cashflows which we might be more concerned about, more carefully than later cashflows which are collected into the larger buckets. The prices are found using the quadratic program (QP) described in the Appendix A. The prices when the economy is in state  $G$  are found by stripping 1,015 coupon bonds using a set of ten market prices for ten dates in 2005. The results for 1 January 2005 are shown in Table 8. We use bonds of all ratings with a spread of maturity dates across all buckets. It is preferable to include a coupon bond maturing in each bucket. The profile of these bonds is shown in Table 7. Zero coupon bond prices when the economy is in state  $B$  are calculated from 1,034 coupon bonds using prices in 2002. The profile for these bonds at 1 January 2002 is shown in Table 9 and the results of the stripping in Table 10.

Table 7: Profile of coupon bonds used for stripping when the economy is in state  $G$

Time to maturity	Treasury	AAA	AA	A	BBB	BB	B	CCC/C	TOTAL
$\leq 1$ Month	1	1	2	0	0	2	4	0	10
$< 1$ Month and $\leq 3$ Months	3	4	4	0	4	6	1	2	24
$< 3$ Months and $\leq 6$ Months	7	6	0	0	0	2	5	2	22
$< 6$ Months and $\leq 1$ Year	10	4	10	15	13	7	5	7	71
$< 1$ Year and $\leq 2$ Years	22	15	15	15	15	13	13	1	109
$< 2$ Years and $\leq 3$ Years	11	10	8	13	9	5	5	6	67
$< 3$ Years and $\leq 5$ Years	28	20	19	19	21	24	21	23	175
$< 5$ Years and $\leq 7$ Years	9	24	12	17	17	22	20	22	143
$< 7$ Years and $\leq 10$ Years	16	18	28	24	25	23	31	15	180
$< 10$ Years and $\leq 20$ Years	25	27	18	21	13	10	12	8	134
$< 20$ Years and $\leq 25$ Years	12	15	10	5	6	14	14	4	80
OVERALL	144	144	126	129	123	128	131	90	1015

Table 8: Zero coupon bond price when the economy is in state  $G$

Time to maturity	Treasury	AAA	AA	A	BBB	BB	B	C
0 Month	1	1	1	1	1	1	1	1
1 Month	0.97212	0.97212	0.97212	0.97212	0.96663	0.93254	0.88168	0.83823
3 Months	0.97131	0.97131	0.97131	0.97131	0.96582	0.93013	0.87927	0.83582
6 Months	0.94399	0.94399	0.94399	0.94399	0.93849	0.90281	0.85195	0.80850
1 Year	0.90173	0.90117	0.90117	0.90117	0.89491	0.83122	0.78036	0.73691
2 Years	0.81344	0.81287	0.81287	0.81287	0.80662	0.74292	0.69206	0.64861
3 Years	0.66322	0.66266	0.66266	0.66266	0.65640	0.59271	0.54184	0.49840
5 Years	0.52655	0.52599	0.52599	0.52599	0.51973	0.45604	0.40517	0.36172
7 Years	0.45454	0.45398	0.45398	0.45398	0.44772	0.38403	0.33316	0.28972
10 Years	0.24624	0.24568	0.24568	0.24568	0.23942	0.17573	0.12486	0.08142
20 Years	0.18131	0.18074	0.18074	0.18074	0.17449	0.11079	0.05993	0.01648

Table 9: Profile of coupon bonds used for stripping when the economy is in state  $B$ 

Time to maturity	Treasury	AAA	AA	A	BBB	BB	B	CCC/C	TOTAL
$\leq 1$ Month	2	0	1	1	1	0	0	0	5
$< 1$ Month and $\leq 3$ Months	5	1	2	1	4	1	3	2	19
$< 3$ Months and $\leq 6$ Months	7	5	5	4	9	6	2	1	39
$< 6$ Months and $\leq 1$ Year	13	9	7	15	6	5	4	3	62
$< 1$ Year and $\leq 2$ Years	26	15	20	20	19	21	15	14	150
$< 2$ Years and $\leq 3$ Years	11	10	6	11	10	4	8	8	68
$< 3$ Years and $\leq 5$ Years	16	22	29	25	23	25	16	16	172
$< 5$ Years and $\leq 7$ Years	10	12	20	14	14	21	28	29	148
$< 7$ Years and $\leq 10$ Years	13	27	20	30	32	26	25	23	196
$< 10$ Years and $\leq 20$ Years	25	16	14	18	20	3	0	2	98
$< 20$ Years and $\leq 25$ Years	10	12	19	13	7	9	2	5	77
OVERALL	138	129	143	152	145	121	103	103	1034

Table 10: Zero coupon bond price when the economy is in state  $B$ 

Time to maturity	Treasury	AAA	AA	A	BBB	BB	B	C
0 Month	1	1	1	1	1	1	1	1
1 Month	0.99999	0.99999	0.99999	0.99999	0.99999	0.99999	0.94260	0.82315
3 Months	0.99916	0.99916	0.99916	0.99916	0.99916	0.99916	0.94177	0.82232
6 Months	0.99749	0.99749	0.99749	0.99749	0.97414	0.92282	0.83770	0.71824
1 Year	0.97590	0.96364	0.94514	0.93937	0.90727	0.85594	0.77082	0.65137
2 Years	0.94229	0.93002	0.91152	0.90575	0.87366	0.82233	0.73721	0.61775
3 Years	0.87697	0.86471	0.84621	0.84044	0.80834	0.75701	0.67189	0.55244
5 Years	0.67221	0.65995	0.64145	0.63568	0.60358	0.55226	0.46714	0.34768
7 Years	0.65903	0.64677	0.62827	0.62250	0.59040	0.53908	0.45396	0.33450
10 Years	0.47752	0.46526	0.44676	0.44099	0.40889	0.35757	0.27245	0.15299
20 Years	0.35699	0.34472	0.32622	0.32045	0.28835	0.23703	0.15191	0.03245

### 3.4 The credit migration process

To calibrate the credit migration process we have used the historical annual transition rates for US Corporate bonds between rating categories for years 1995 - 2005. Typical data for a year is shown in Table 11.

Table 11: Rating Transitions in 2005

	Number of Issuers	AAA	AA	A	BBB	BB	B	CCC/C	D	NR
AAA	60	88.33	11.67	0.00	0.00	0.00	0.00	0.00	0.00	0.00
AA	152	0.00	93.42	3.29	0.00	0.00	0.00	0.00	0.00	3.29
A	505	0.20	0.99	89.50	6.34	0.00	0.00	0.00	0.00	2.97
BBB	753	0.00	0.27	2.39	88.84	4.25	0.66	0.00	0.13	3.45
BB	550	0.00	0.00	0.00	3.64	78.00	9.09	0.36	0.36	8.55
B	696	0.00	0.00	0.14	0.29	5.60	72.27	4.60	2.30	14.80
CCC/C	74	0.00	0.00	0.00	1.35	0.00	18.92	50.00	10.81	18.92

The annual transition matrix when the economy is in state  $G$  is calculated in the following way. We ignore those transitions to the state NR - non-rated and remove these from the data, following Jarrow et al. (1997). To estimate  $p_{jj'}^G(1)$  the total number of transitions from rating  $j$  to  $j'$  over the year, for those years where the economy is classed as  $G$  are considered. Let  $a$  be the number of issuers with rating  $j$  at the start of the relevant years and  $b$  is the total number of issuers who move from state  $j$  to  $j'$ . The probability  $p_{jj'}^G(1)$  is calculated as  $\frac{b}{a}$ .

When the time period being considered is a fraction of a year, the transition matrix is calculated by determining the appropriate root of the annual transition matrix,  $R$ . Using the eigenvalues,  $\lambda_1, \dots, \lambda_M$ , and eigenvectors,  $\mathbf{x}_1, \dots, \mathbf{x}_M$ , we can express  $R$  as  $X\Lambda X^{-1}$ . Here  $X$  is a matrix where the  $i$ th column is the vector  $\mathbf{x}_i$  and  $\Lambda$  is a diagonal matrix where the  $i$ th element is  $\lambda_i$ . The  $n$ th root of  $R$  is  $X\Lambda^*X^{-1}$ , here  $\Lambda^*$  is a diagonal matrix where the  $i$ th element is  $\lambda_i^{\frac{1}{n}}$ . A similar method is outlined in Hull (2003). The physical world transition matrices we derived are shown in Appendix C in Tables 15 to 52.

Recovery rates are not readily available so it is assumed that  $\eta^G = 0.35$  and  $\eta^B = 0.2$ .

Next we compute the risk neutral measure for the credit rating process; the change of measure for ratings transitions is set out in equations (34), (35) and (36). We use the zero coupon bond prices which are derived as per section 3.3 and Appendix A. For calibration we assume that the observed zero coupon bond prices, at a given date when the economy is in state  $e \in S^E$ , are based on investors' belief the economy will remain in this state until the bond matures. Here we outline the method given by Jarrow et al. (1997) in deriving the risk neutral measure given the zero coupon bond prices at a particular date.

The price of a zero coupon bond is its expected discounted value under the pricing measure. We can separate the interest rate and credit migration processes as they are assumed to be independent under the pricing measure assuming the state of the economy does not change. We have

$$(52) \quad Z_t^T(s_t, j, e) = Z_t^T(s_t, 0, e) \left[ \eta^G + (1 - \eta^G) \tilde{P}(\tau^j > T) \right].$$

Then the default probabilities are implied by the zero coupon bond prices,

$$(53) \quad \tilde{P}(\tau^j > T) = \frac{Z_t^T(s_t, j, G) - \eta^G}{(1 - \eta^G)}.$$

That is the risk neutral probabilities over  $(0, T_n]$  are known. As the process is Markov, if probabilities over  $(0, T_{n-1}]$  and probabilities over  $(0, T_n]$  are known we can derive probabilities over  $(T_{n-1}, T_n]$ . Risk neutral probabilities over  $(0, T_0]$  can be found from zero coupon bond prices for bond maturing at time  $T_0$ . We can derive risk neutral probabilities over all timeperiods recursively.

The above procedure cannot ensure that the risk neutral probabilities are non-negative and sum to unity. We overcome this difficulty by formulating the calibration method as a series of quadratic optimisation problems with candidate solutions lying on the unit hyperplane. Each problem finds the risk neutral probabilities which minimise the squared error between the observed prices and expected prices of the zero coupon bonds for a particular maturity  $T_n$ . The QP formulation is given in Appendix B. Wei (2000) and Albanese & Chen (2006) adopt similar approaches. We also use zero coupon bond prices at several different dates when formulating this QP. Thus, we incorporate more information from the larger set of observations in deriving the values. We use prices at the start of the first ten months in 2005 when calibrating for risk adjustment factors when the economy is in state  $G$ . Similarly, we use ten dates in 2002 when the economy is in state  $B$ .

The risk neutral probabilities for the calibration process are summarised in a series of Tables shown in Appendix C. Tables 31 and 32 contain the risk adjustment factors that which connect the risk neutral and real world measures.

## 4 Evaluation of the bond portfolio

Our model can be used to calculate the volatility and risk measures for an individual bond or those for a bond portfolio. The most popular risk measures are (symmetric) variance and (asymmetric) VaR (Hull 2003) and CVaR (Urayasev 2002).

We calculate the VaR and CVaR for a number of bond portfolios one year ahead. Hull (2003) and Dowd (1998) note that such a time period is appropriate when considering credit risk. In particular, Dowd comments that the period of time we should consider depends on the holding period of the asset. Bond portfolios are often held by pension funds, which do not trade daily and hold assets over longer time horizons. Several other papers consider a year ahead when investigating credit risk, (Jobst et al. 2006). Table 12 contains the bond portfolios which have been used in the computational study.

Table 12: Portfolios Considered in Computational Study

Portfolio	Bonds	Weights %
P1	Treasury bond, maturity 20 years	100.00%
P2	C rated bond, maturity 20 years	100.00%
P3	Treasury bond, maturity 20 years	50.00%
	AA rated bond, maturity 20 years	50.00%
P4	Treasury bond, maturity 20 years	33.34%
	AA rated bond, maturity 20 years	33.33%
	BBB rated bond, maturity 20 years	33.33%

We wish to evaluate the risk exposure of an individual bond or that of a portfolio starting in a good economic state and a bad economic state.

#### 4.1 Pricing and yield curve simulation

In simulating and pricing the bonds the following assumptions are made. We assume that the economy process changes state annually with the first change occurring halfway through the year. The interest rate process jumps daily. Ratings are assumed to change at the times in the set  $SR = \{\frac{1}{12}, \frac{3}{12}, \frac{6}{12}, 1, 2, 3, 5, 7, 10, 20\}$ , which coincide with the maturity dates of the zero coupon bonds.

For any particular realisation of the world all bonds are influenced by the same states of the economy and same underlying interest rates. These two processes are simulated together. The state of the economy at the start of the investigation is known. We pick from a uniform distribution over (0,1) to determine whether the economy changes or stays the same halfway through the year. We take the square root of the annual transition matrix to determine the probabilities of transitions for the economy process over half a year and use these and the random number that has been picked to determine the new state of the economy. We choose 250 further random numbers on the uniform distribution over (0,1) to simulate the interest rate process. The probabilities used to determine whether the process has jumped up or down depend on the state of the economy at the start of the year and midway through it. The size of the jumps also depend on the realisation of the economy. As the process is recombining the order of the upward and downward jumps do not matter. We are able to calculate the yield curve one year forward given the number of upward and downward jumps in each state of the economy.

Each bond is simulated separately to give the bond's new rating at the end of the year. For each simulation we take the realisation of the economy from the first part, then the probabilities of changing rating class over the year are known. We pick one uniform random number to derive a new rating for the bond.

For each simulation we have an interest rate curve and we know the state of the economy and each bond's new rating. We price each bond individually and we note from the pricing formulae (37) that we need to determine  $A_0^{T_n}(j, e)$  for each simulation. Any bond can only go to a finite number of states, so rather than recalculating  $A_0^{T_n}(j, e)$  several times we calculate all possible values and look up the appropriate value for each simulation and discount it by the interest rate.

## 4.2 Results

In order to evaluate the effect of the good and bad economies we have carried out two sets of experiments, one assuming that the economy starts in state  $G$  and the other in state  $B$ . When the economy starts in state  $G$  we use the initial yield curve as at 30 December 2005 and when it starts in state  $B$  we use the curve as at 31 December 2002. The VaR and CVaR are calculated at the 95 % level and we use 1,000 simulations in both experiments.

The initial prices have been calculated using the appropriate yield curves and the derived risk adjustment factors. The expected return is the annualised return implied by the initial price.

The portfolio value under each simulation is calculated as the weighted value of the component bonds prices. The VaR is then calculated as the 5% lower quantile. For the CVaR we take the average value of those simulated portfolios where the value is less than or equal to the VaR. We report the VaR and CVaR as the loss as a percentage of initial portfolio value.

The results of the experiment when the state of the economy is  $G$  is reported in Table 13 and those when the economy is bad in Table 14.

Table 13: Simulation results when the economy is in state  $G$

Portfolio	Bonds	Weights %	Expected Return over a year	VaR over a year (Loss Amount)	CVaR over a year (Loss Amount)
P1	Treasury bond, maturity 20 years	100.00%	4.61%	1.60%	1.78%
P2	C rated bond, maturity 20 years	100.00%	10.86%	13.22%	13.47%
P3	Treasury bond, maturity 20 years	50.00%	4.92%	4.49%	7.13%
	AA rated bond, maturity 20 years	50.00%			
P4	Treasury bond, maturity 20 years	33.34%	5.52%	9.58%	11.17%
	AA rated bond, maturity 20 years	33.33%			
	BBB rated bond, maturity 20 years	33.33%			

Table 14: Simulation results when the economy is in state  $B$

Portfolio	Bonds	Weights %	Expected Return over a year	VaR over a year (Loss Amount)	CVaR over a year (Loss Amount)
P1	Treasury bond, maturity 20 years	100.00%	4.83%	0.82%	1.16%
P2	C rated bond, maturity 20 years	100.00%	11.29%	9.37%	9.81%
P3	Treasury bond, maturity 20 years	50.00%	5.35%	5.04%	9.12%
	AA rated bond, maturity 20 years	50.00%			
P4	Treasury bond, maturity 20 years	33.34%	6.20%	5.98%	9.73%
	AA rated bond, maturity 20 years	33.33%			
	BBB rated bond, maturity 20 years	33.33%			

## 5 Concluding remarks

We have described a regime switching Markovian pricing model for credit risky bonds and calibrated this model using historical market data. Through the introduction of the three Markov chains we integrate market and credit risk under consideration of the state of the economy. We determine the physical and pricing measures and describe the relationship between them. In particular our model extends the approach of Thomas et al. by considering the entire default-free yield curve with appropriate restrictions to exclude arbitrage across different maturities. A QP is used to discover the underlying zero coupon bond prices from the market prices of coupon bonds; the constraints of the QP remove anomalous pricing. The risk neutral measure for the credit migration process is computed using a second QP model in which constraints ensure probabilities are non-negative and sum to unity.

The model has been used to simulate the distribution of credit risky bonds portfolios and thus calculate risk measures. A further extension to the work would be to simulate the risk process over time.

Markets for credit derivative products, as described by the indices iTraxx and CDX, are much more liquid than the underlying bond markets. These might provide a better way for calibrating credit models. (Albanese, Chen, Dalessandro & Vidler (2006))

We present a descriptive model and we only price the future portfolio and quantify its risk exposure but do not investigate the management of the credit risk of the portfolio. That is we do not make any decisions about what the optimal portfolio to hold might be. Jobst (2002) and Jobst, Mitra & Zenios (2006) investigate the use of multi-stage stochastic optimisation decision models for managing portfolios with credit sensitive and interest rate sensitive assets. However, our Markovian regime switching model can be used to generate scenarios which describe asset price dynamics. These scenarios can be used for a stochastic programming ex-ante decision model and for out of sample simulation to study the volatility and risk behaviour of bond portfolios.

## 6 Acknowledgements

We would also like to thank Prof. Ziemba for his careful reading of the paper and helpful comments. Jong Park from Standard and Poors kindly provided data on transition rates. Leela Mitra is sponsored by an EPSRC grant and CASE studentship from OptiRisk Systems Ltd.

## References

- [1] Albanese, C. & Chen, O. 2006. "Implied migration rates from credit barrier models", *Journal of Banking and Finance* Vol 30, Issue 2, p.607-626.
- [2] Albanese, C., Chen, O., Dalessandro, A. & Vidler, A. 2006. "Dynamic credit correlation modeling" *Working Paper on www.level3finance.com*.
- [3] Anderson, R. & Sundaresan, S. 1996. "Design and valuation of debt contracts" *Review of Financial Studies* Vol 9, p.37-68.
- [4] Bielecki, T.R. & Rutkowski, M. 2002. *Credit risk: modeling, valuation and hedging* Springer Finance 2002.
- [5] Black, F., & Cox, J., 1976. "Valuing corporate securities: Some effects of bond indenture provisions" *Journal of Finance* Vol 31, p.351-367.
- [6] Black, F., Derman, E. & Toy, W., 1990. "A one-factor model of interest rates and its application to treasury bond options" *Financial Analysts Journal* Vol 46(1), p.33-39.
- [7] Cathcart, L. & El-Jahel, L. 1998. "Valuation of defaultable bonds" *Journal of Fixed Income* Vol 8(1), p.65-78.
- [8] Cox, J., Ingersoll & J.E., Ross, S.A. 1985. "A theory of the term structure of interest rates" *Econometrica* Vol 53, p.385-407.
- [9] Duffie, D., & Lando, D. 2001. "The term structures of credit spreads with incomplete accounting information" *Econometrica* Vol 69, p.633-664.
- [10] Duffie, D., & Singleton, K. 1999. "Modeling term structures of defaultable bonds" *Review of Financial Studies* Vol 12(4), p.687-720.
- [11] Geske, R. 1977. "The valuation of corporate liabilities as compound options" *Journal of Financial Quantitative Analysis* Vol 12, p.541-552.
- [12] Geske, R. 1979. "The valuation of compound options" *Journal of Financial Economics* Vol 7, p.63-81.
- [13] Giampieri, G., Davis, M., & Crowder M. 2005. "A hidden markov model of default interaction" *Working Paper, Imperial College, London*.
- [14] Di Graziano, G. & Rogers, L.C.G. 2005. "A new approach to the modelling and pricing of correlation credit derivatives" *Working Paper, Cambridge University*.
- [15] Heath, D., Jarrow, R.A., & Morton, A. 1992. "Bond pricing and term structure of interest rates: A new methodology for contingent claims valuation" *Econometrica* Vol 60(1), p.77-105.
- [16] Harrison J.M & Kreps 1979. "Martingales and arbitrage in multiperiod securities markets" *Journal of Economic Theory* Vol 20, p.381-408.
- [17] Harrison, J.M. & Pliska S.R. 1981. "Martingales and stochastic integrals in the theory of continuous trading" *Stochastic Process Applications* Vol 11, p.215-260.
- [18] Hull, J.C. 2003. *Options, futures and other derivatives* Prentice Hall 2003.
- [19] Jarrow, R.A., & Turnbull, S.M. 1995. "Pricing derivatives on financial securities subject to credit risk" *The Journal of Finance* Vol 50, p.53-85.
- [20] Jarrow, R.A., Lando, D. & Turnbull, S.M., 1997. "A markov model for the term structure of credit risk spreads" *The Review of Financial Studies* Vol 10, Issue 2, p.481-523.
- [21] Jarrow, R.A., 2002. *Modeling fixed-income securities and interest rate options* Stanford Economics and Finance 2002

- [22] Jones, E., Mason, S. & Rosenfeld, E. 1984. "Contingent claim analysis of corporate capital structures: An empirical investigation" *Journal of Finance* Vol 39, p.611-625.
- [23] Jobst, N.J., 2002. "On credit risk modelling, measurement and optimisation" *PhD Thesis, Brunel University*.
- [24] Jobst, N.J., Mitra, G. & Zenios, S.A. 2006. "Integrating market and credit risk: A simulation and optimisation perspective" *Journal of Banking & Finance* Vol 30, Issue 2 p.717-742.
- [25] Kijima, M., & Komoribayashi, K. 1998. "A markov chain model for valuing credit risk derivatives" *Journal of Derivatives* Vol 6(1), p.97-108.
- [26] Kim, I.J., Ramaswamy, & K. Sundaresan, S. 1993. "The valuation of corporate fixed income securities" *Working Paper, Wharton School, University of Pennsylvania*.
- [27] Lando, D., 1999. "Some elements of rating-based credit risk modeling" *Working Paper, University of Copenhagen*.
- [28] Leland, H.E., 1994. "Corporate debt value, bond covenants, and optimal capital structure" *Journal of Finance* Vol 49 , p.1213-1252.
- [29] Leland, H.E. & Toft, K. 1996. "Optimal capital structure, endogenous bankruptcy and the term structure of credit spreads" *Journal of Finance* Vol 51, p.981-1019.
- [30] Litterman, R. & Iben, T. 1991. "Corporate bond valuation and the term structure of credit spreads" *The Journal of Portfolio Management* Vol 17 , p.52-64.
- [31] Longstaff, F.A. & Schwartz, E.S. 1995. "A simple approach to valuing risky fixed and floating rate debt" *Journal of Finance* Vol 50 , p.789-819.
- [32] Mamon, R.S. & Rodrigo, M.R., 2005. "Explicit solutions to european options in a regime switching economy" *Operations Research Letters* Vol 33(6), p.581-586.
- [33] Madan, D. & Unal, H. 1998. "Pricing the risks of default" *Review of Derivatives Research* Vol 2, p.121-160.
- [34] Madan, D. & Unal, H. 2000. "A two factor hazard rate model for pricing risky debt and the term structure of credit spreads" *Journal of Financial and Quantitative Analysis* Vol 35, p.43-65.
- [35] Mella-Barral, P. & Perraudin, W., 1997. "Strategic debt service" *Journal of Finance* Vol 52, p.531-556.
- [36] Merton R.C. 1974. "On the pricing of corporate debt: The risk structure of interest rates" *Journal of Finance* Vol 29, p.449-470.
- [37] Pliska, S.R., 1997. *Introduction to Mathematical Finance* Blackwell, Oxford 1997.
- [38] Rockafeller, R.T., & Uraysev, S. 2002. "Conditional Value at Risk for general loss distributions" *Journal of Banking and Finance* Vol 26, p.1443 - 1471.
- [39] Rogers L.C.G. 1999. "Modelling credit risk" *Working Paper, University of Bath*.
- [40] Shimko, D., Tejima, N. & van Deventer, D.R. 1993. "The pricing of risky debt when interest rates are stochastic" *Journal of Fixed Income* Vol 3, p.58-65.
- [41] Szatschneider, W. 2000. "CIR model in financial markets" *Technical report, Anahuac University*.
- [42] Thomas, L.C., Allen, D.E. and Morkel-Kingsbury, N., 2002. "A hidden Markov chain model for the term structure of bond credit spreads" *International Review of Financial Analysis* Vol 11, Issue 3, p.311-329.
- [43] Thomas, L.C., Allen, D.E. and Zheng, H., 2000. "Stripping coupons with linear programming" *The Journal of Fixed Income* Vol 10, Issue 2, p.80.

- [44] Wang, D.F. 1999. “Pricing defaultable debt: Some exact results” *International Review of Financial Analysis* Vol 11, Issue 3, p.311-329.
- [45] Wei, J. 2003. “A multi-Factor, markov chain model for credit migrations and credit spreads” *Journal of International Money and Finance* Vol 22, Issue 5, p.709-735.
- [46] Vasicek, O. 1977. “An equilibrium characterisation of the term structure” *Journal of Financial Economics* Vol 5, p177-188.
- [47] Vasicek, O. 1984. “Credit valuation” *Technical Report, KMV Corporation* p177-188.
- [48] Zhou, C. 1996. “A jump-diffusion approach to modeling credit risk and valuing defaultable securities” *Technical Report, Federal Reserve Board, Washington*.
- [49] Schonbucher, P.J. 1996. “The term structure of defaultable bond prices” *Technical Report, University of Bonn*.

## A Stripping coupons using quadratic programming QP

A coupon bond can be viewed as a portfolio of zero coupon bonds. So if we assume that markets are arbitrage-free the price of a bond  $b$  at time 0 would be

$$p_b = \sum_{t=1}^T c_b(t) z_{r(b)}(t).$$

Here bond  $b$  pays coupons of  $a$  semi-annually at times  $t = 1, \dots, T$  and at maturity the face value of  $F$  is returned. Let  $c_b(t)$  represent a cash flow at time  $t$  where  $c_b(t) = a$  for  $t \neq T$  and  $c_b(T) = a + F$ . Also  $r(b)$  gives the rating of bond  $b$  and  $z_{r(b)}(t)$  is the zero coupon bond price for zero coupon bond of rating  $r(b)$  which matures at time  $t$ . Zero coupon bond prices act as discount factors.

Models for term structure of credit risk free and credit risky bonds describe zero coupon bond prices or equivalently spot rates or forward rates. However, mainly coupon bonds are available in the market. Only a relatively small number of STRIPS are traded and as a result these are not very liquid instruments and their returns include a premium for liquidity. So we need to determine the underlying zero coupon bond prices from the observed market prices of coupon bonds before we can calibrate our model.

Traditional methods for doing this include “bootstrapping”, linear regression and the use of cubic splines. As can be seen above a coupon bond’s price depends on its rating, maturity and coupon. Jarrow, Lando and Turnbull (1997) bucket bonds by rating and maturity and then calculate the average prices and coupons for each bucket. They then solve a system of triangular equations to derive the zero coupon bond prices. This is known as bootstrapping. As long as one coupon bond matures in each time period this gives a unique set of prices. Jarrow et al., however, find that anomalous pricings or mispricings occur. In particular, bonds with worse ratings do not always have prices below bonds with higher ratings as would be expected in an arbitrage-free economy.

Allen, Thomas and Zheng (2000) use linear programming to derive zero coupon bond prices. To do this they find the zero coupon bond prices that minimise the absolute pricing error. The distinguishing feature of their approach is the introduction of constraints which remove mispricing. They also recognise that a bond’s price is affected by its relative liquidity and so they incorporate the size of a bond’s issue in their linear program (LP). We use a similar formulation to Allen et al. but instead we use quadratic programming (QP) to minimise the squared pricing errors. This is equivalent to a constrained regression. We compare the results of QP and LP to investigate the difference between the methods. In particular, we consider an “efficient frontier” showing the trade-off between the two objectives.

As before, consider the set of rating classes  $S^R = \{0, 1, \dots, M\}$ , where 0 represents default-free government bonds and  $M$  bankruptcy. We consider the time set  $S^T = \{0, \dots, T^*\}$ . We have a set of  $N$  bonds  $S^B =$

$\{1, \dots, N\}$ , where bond  $b$  has a current market price of  $v_b$ , rating given by  $r(b) \in S^R$  and pays cash flow  $c_b(t)$  at time  $t$ .  $z_k(t)$  is the unknown price of a zero coupon bond with rating  $k$  which pays 1 at time  $t$ . Finally let  $o_b$  represent the pricing error “over” and  $u_b$  the pricing error “under”. We solve the following quadratic program.

$$(54) \quad \text{Minimise} \quad \sum_{b \in S^B} (o_b + u_b)^2$$

subject to

$$(55) \quad v_b + o_b = \sum_{t=1}^T c_b(t) z_{r(b)}(t) + u_b \quad \forall b \in S^B$$

$$(56) \quad z_0(t) \geq z_0(t+1)(1+m(t)) \quad \forall b \in S^B$$

$$(57) \quad z_k(t+1) - z_{k+1}(t+1) \geq z_k(t) - z_{k+1}(t) \quad \forall t \in S^T, \quad \forall k \in \{0, \dots, M-1\}$$

As already stated we are minimising the squared errors. We expect the market price of the bond to equal its cash flows discounted by appropriate zero coupon bond prices with some error term. This is shown in (55). The next constraint ensures that as the term to maturity of default free zero coupon bonds increases their prices decrease. Finally (57) requires zero coupon bond prices for ratings 1 to  $M-1$  are also decreasing functions with respect to time to maturity. It also forces bonds with worse ratings to have lower prices.

To simplify the calculation and reduce the number of cash flow dates it is assumed all bonds pay their coupons on the same dates. The dates considered are  $\frac{1}{12}, \frac{3}{12}, \frac{6}{12}, 1, 2, 3, 5, 7, 10, 20$  years from the calculation date. The cash flows are adjusted for the change in timing of payments. Cash flows that fall between these dates are split proportionally to the time intervals between the start and end of the buckets.

Figure 4 shows the results for 140 bonds. Their market prices are as at 24 February 2002.

Time	rat0	rat1	rat2	rat3	rat4	rat5	rat6	rat7	rat8	rat9
t0	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
t1	0.990099	0.990099	0.990099	0.990099	0.990099	0.983962	0.979021	0.979021	0.979021	0.979021
t2	0.980296	0.980296	0.980296	0.980296	0.96019	0.933766	0.891500	0.891500	0.891500	0.891050
t3	0.970590	0.970590	0.967773	0.958247	0.931388	0.904964	0.814784	0.814784	0.814784	0.814784
t4	0.952417	0.952417	0.949600	0.940074	0.913215	0.886791	0.796611	0.796611	0.796611	0.796611
t5	0.942987	0.942987	0.940170	0.930644	0.903785	0.877361	0.787181	0.787181	0.787181	0.787181
t6	0.902465	0.902465	0.898283	0.888756	0.861897	0.835473	0.745293	0.745293	0.745293	0.745293
t7	0.838184	0.838184	0.834001	0.824474	0.797615	0.771191	0.676488	0.676488	0.676488	0.676488
t8	0.829885	0.829885	0.825702	0.816176	0.789317	0.762893	0.668189	0.668189	0.641795	0.641795
t9	0.821668	0.821668	0.817486	0.807959	0.781100	0.754676	0.659973	0.659973	0.633578	0.633578
t10	0.813533	0.813533	0.809350	0.799824	0.772965	0.746541	0.651837	0.651837	0.625443	0.625443
t11	0.763743	0.763743	0.759560	0.750033	0.723174	0.696750	0.591850	0.591850	0.565455	0.565455
t12	0.756181	0.756181	0.751998	0.742472	0.715613	0.689189	0.584288	0.584288	0.475865	0.475865
t13	0.748694	0.748694	0.744511	0.734985	0.708126	0.681702	0.576801	0.576801	0.468379	0.468379
t14	0.703403	0.703403	0.699221	0.689694	0.662835	0.636411	0.531510	0.531510	0.423088	0.423088
t15	0.661618	0.661618	0.657436	0.647909	0.621050	0.594626	0.489726	0.489726	0.381303	0.381303

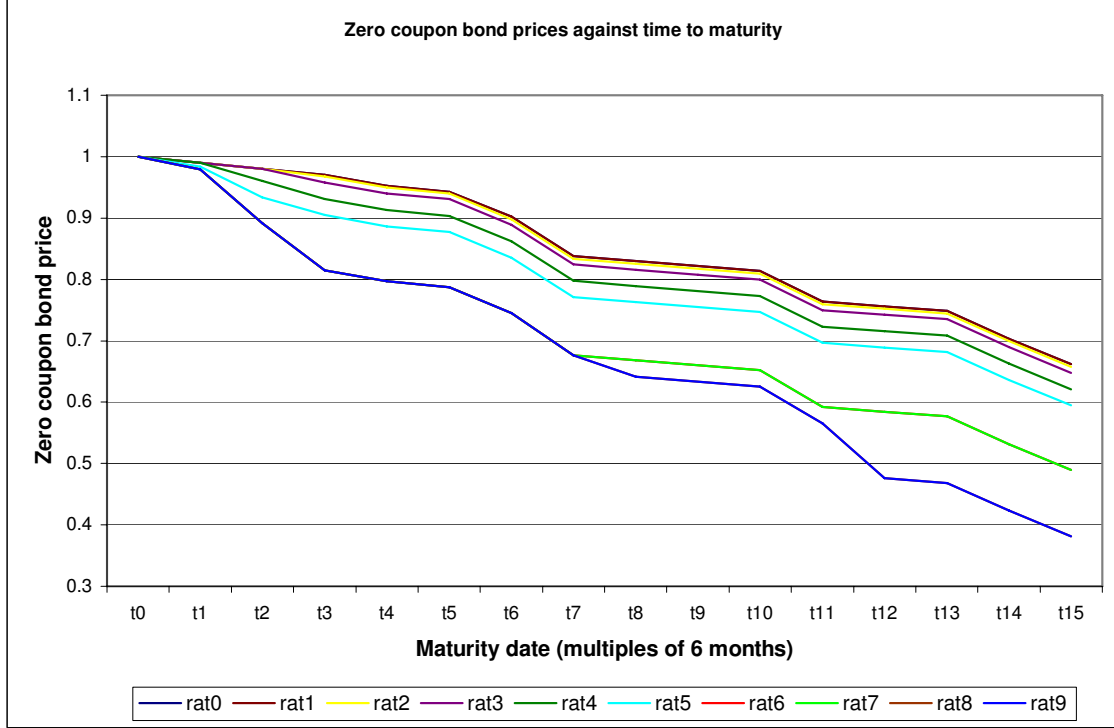


Figure 4: Results of QP with arbitrage constraint

We notice that the zero coupon bond prices for Treasury and AAA rated bonds are identical. They are also identical for B and CCC. Generally the curves are very similar for bonds of similar ratings. To investigate this we change constraint (57) so that the prices are still required to be decreasing functions of time to maturity, however, bonds which have worse ratings are no longer required to have lower prices than those with better ratings. So we solve

$$(58) \quad \text{Minimise} \quad \sum_{b \in S^B} (o_b + u_b)^2$$

subject to

$$(59) \quad v_b + o_b = \sum_{t=1}^T c_b(t) z_{r(b)}(t) + u_b \quad \forall b \in S^B$$

$$(60) \quad z_0(t) \geq z_0(t+1)(1+m(t)) \quad \forall b \in S^B$$

$$(61) \quad z_0(t+1) - z_{k+1}(t+1) \geq z_0(t) - z_{k+1}(t) \quad \forall t \in S^T \\ \forall k \in \{0, \dots, M-1\}.$$

We display the results for the same data set in Figure 5.

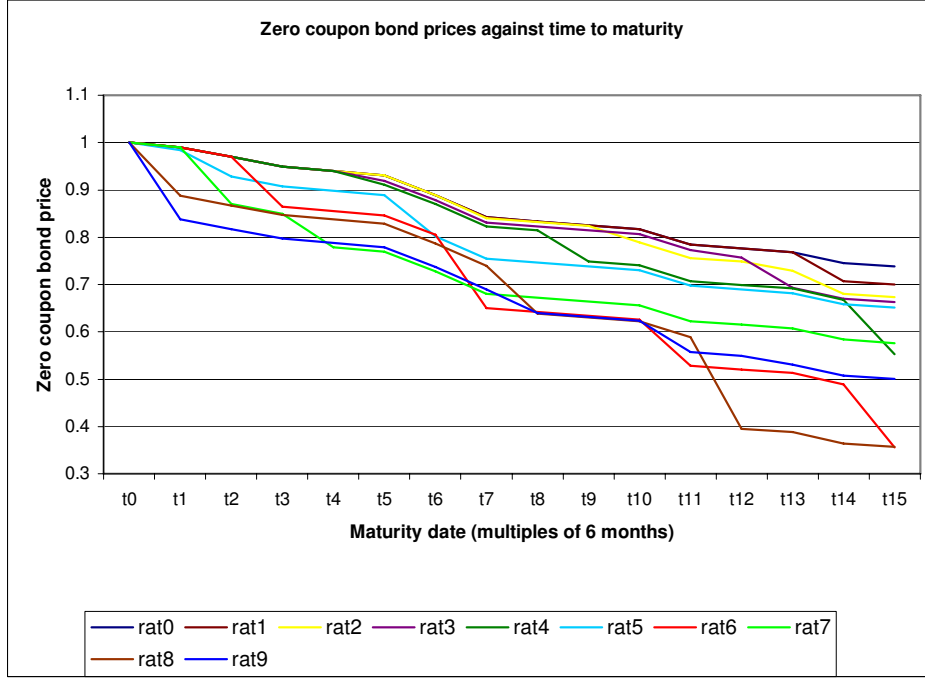


Figure 5: Anomalous results of QP without arbitrage constraints

There are many zero coupon bond prices which will price the coupon bonds close to their observed market prices. We are however more interested in those solutions which will result in prices that show properties which we expect to observe in the market. The introduction of appropriate constraints allows us to do this.

We were interested in the difference in the results of the LP and QP. So we reformulate the original QP with an additional constraint that fixes the level of the absolute deviation.

$$(62) \quad \text{Minimise} \quad \sum_{b \in S^B} (o_b + u_b)^2$$

subject to

$$(63) \quad v_b + o_b = \sum_{t=1}^T c_b(t) z_{r(b)}(t) + u_b \quad \forall b \in S^B$$

$$(64) \quad z_0(t) \geq z_0(t+1)(1 + m(t)) \quad \forall b \in S^B$$

$$(65) \quad z_0(t+1) - z_{k+1}(t+1) \geq z_0(t) - z_{k+1}(t) \quad \forall t \in S^T \\ \forall k \in \{0, \dots, M-1\}$$

$$(66) \quad \sum_{b \in S^B} o_b + u_b = D.$$

This QP problem is solved for a number of values of  $D$  that lie between the minimum absolute error that can be found from the LP and the absolute error given when using the optimal solution for the QP. This allows us to plot an “efficient frontier” of absolute pricing error against squared pricing errors. If we wish to have results that give low absolute pricing errors and squared pricing errors we might use result from point A or B.

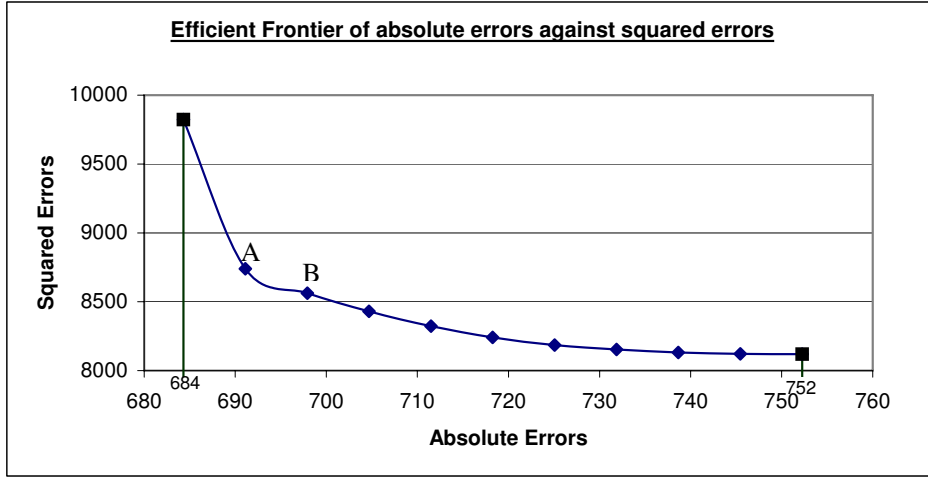


Figure 6: Efficient Frontier showing trade-off between minimising squared errors and absolute errors

## B Credit migration process calibration

We calibrate the risk neutral probability measure by using the migration rates implied by the market zero coupon bond prices, empirical credit migration rates and assumed default rate. These implied rates may, however, be negative. So we minimise the difference between the calculated bond prices and observed prices, introducing the constraint that the probabilities are non-negative, less than or equal to 1 and sum to unity. The method outlined in section 3.4 considers calibration using observed prices at one date. The following method incorporates the use of several observation dates.

We define the following sets

- TP =  $\{T_0, \dots, T_k\}$  timepoints considered,
- SB =  $\{T_0, \dots, T_{k-1}\}$  timepoints denoting the start of the buckets,
- EB =  $\{T_1, \dots, T_k\}$  timepoints denoting the end of the buckets,
- CS =  $\{1, \dots, M\}$  all credit states excluding Treasury bonds,
- CS2 =  $\{1, \dots, M - 1\}$  all credit states excluding Treasury bonds and the default state,
- OBS =  $\{1, \dots, N\}$  observation times of prices.

We first find the risk neutral probabilities over  $(T_0, T_1)$  using the QP initial credit fit. Then we determine the risk neutral probabilities over  $(T_n, T_{n+1})$ , recursively using the QP recursive credit fit.

### B.1 QP initial credit fit

The decision variables are the risk adjustment factors,  $\pi(T_0, j)$   $j \in \text{CS2}$ ; the risk neutral probabilities  $\tilde{p}_{j, j'}(T_0)$ ,  $j, j' \in \text{CS2}$  and the zero coupon bond prices implied by the calibrated model. We denote these prices by  $ZC_{T_0}^{T_k}(j, n)$  the zero coupon bond price calculated by the calibrated model at time  $T_0$  for a bond maturing at  $T_k$  with rating  $j$ , when the price of a Treasury bond is the price observed at  $n$ ,  $n \in \text{OBS}$ . To simplify the notation in this section, we drop the state of the economy and state of the ratings' superscripts. At time  $t = 0$  we see that  $T_0 = t = 0$  the interest rate process is in state  $s_t = 0$ , so we also drop the use of  $s_t$ .

We have the following items as data inputs, that is, parameters to the problem.  $Z_{T_0}^{T_1}(j, n)$  are the observed zero coupon bond prices for  $j \in CS$ ,  $n \in OBS$  and  $Z_{T_0}^{T_1}(0, n)$  are observed prices for Treasury bonds. The physical world transition rates over  $(T_0, T_1]$  are also data items,  $p_{j j'}(T_1 - T_0)$ ,  $j, j' \in CS2$ .

To find  $\pi(T_0, j)$  we solve the following QP,

$$(67) \quad \text{Minimise} \quad \sum_{n \in OBS} \sum_{j \in CS2} (Z_{T_0}^{T_1}(j, n) - ZC_{T_0}^{T_1}(j, n))^2$$

subject to

$$(68) \quad ZC_{T_0}^{T_1}(j, n) = Z_{T_0}^{T_1}(0, n)[1 + \tilde{p}_{j M}(T_0) (\eta - 1)] \quad \forall j \in CS2 \quad \forall n \in OBS$$

$$(69) \quad \tilde{p}_{j j'}(T_0) \geq 0 \quad \forall j j' \in CS$$

$$(70) \quad \tilde{p}_{j j'}(T_0) \leq 1 \quad \forall j j' \in CS$$

$$(71) \quad \sum_{j \in CS} \tilde{p}_{j j'}(T_0) = 1 \quad \forall j j' \in CS$$

$$(72) \quad \tilde{p}_{j j'}(T_0) = \pi(T_0, j)p_{j j'}(T_1 - T_0) \quad \forall j j' \in CS2.$$

## B.2 QP recursive credit fit

We then solve the following QP recursively, to find  $\pi(T_n, j)$ , for  $n \in \{1, \dots, k\}$ . We introduce  $\tilde{p}_{j j'}(T_0, T_n)$  to represent the risk neutral probability of going from state  $j$  to  $j'$  in time period  $(T_0, T_n]$ .

Consider the time period  $(T_n, T_{n+1}]$ . The decision variables are the risk adjustment factors  $\pi(T_n, j)$ , the risk neutral probabilities  $\tilde{p}_{j j'}(T_0, T_{n-1})$  and the zero coupon bond prices,  $ZC_{T_0}^{T_n}(j)$  implied by the calibrated model.

The parameters are the observed bond prices  $Z_{T_0}^{T_n}(j, n)$  for  $j \in CS2$   $n \in OBS$ , observed prices for Treasury bonds  $Z_{T_0}^{T_1}(0, n)$  and physical world transition rates  $p_{j j'}(T_{n+1} - T_n)$ .

To find  $\pi(T_n, j)$  we solve the following QP problem,

$$(73) \quad \text{Minimise} \quad \sum_{n \in OBS} \sum_{j \in CS2} (Z_{T_0}^{T_n}(j, n) - ZC_{T_0}^{T_n}(j, n))^2$$

subject to

$$(74) \quad ZC_{T_0}^{T_n}(j) = Z_{T_0}^{T_n}(0, n)[1 + \tilde{p}_{j M}(T_0, T_n) (\eta - 1)] \quad \forall j \in CS2 \quad \forall n \in OBS$$

$$(75) \quad \tilde{p}_{j j'}(T_n) \geq 0 \quad \forall j j' \in CS$$

$$(76) \quad \tilde{p}_{j j'}(T_n) \leq 1 \quad \forall j j' \in CS$$

$$(77) \quad \sum_{j \in CS} \tilde{p}_{j j'}(T_n) = 1 \quad \forall j j' \in CS$$

$$(78) \quad \tilde{p}_{j j'}(T_n) = \pi(T_n, j)p_{j j'}(T_n - T_0) \quad \forall j j' \in CS2$$

$$(79) \quad \tilde{p}_{j M}(T_0, T_n) = \sum_{k \in CS} \tilde{p}_{j M}(T_0, T_{n-1})\tilde{p}_{j j'}(T_n) \quad \forall j j' \in CS$$









Table 30: Rating transitions over ten years economy state B

	AAA	AA	A	BBB	BB	B	CCC/C	D
AAA	59.121	20.746	13.655	4.649	0.772	0.444	0.100	0.512
AA	1.889	27.122	39.480	21.017	4.441	2.210	0.507	3.334
A	0.149	5.298	39.902	34.879	9.019	3.861	0.855	6.038
BBB	0.069	1.218	10.235	45.017	18.541	9.388	1.739	13.794
BB	0.175	0.337	2.276	11.118	24.257	20.025	3.605	38.208
B	0.028	0.064	0.562	2.434	8.656	16.761	3.319	68.176
CCC/C	0.006	0.040	0.356	0.622	1.704	3.175	0.688	93.408
D	0.000	0.000	0.000	0.000	0.000	0.000	0.000	100.000

## C.2 Pricing measure

Table 31: Risk adjustment factors economy state  $G$ 

	0 Months	1 Month	3 Months	6 Months	1 Year	2 Years	3 Years	5 Years	7 Years	10 Years
	-1 Month	-3 Months	-6 Months	-1 Year	-2 Years	-3 Years	-5 Years	-7 Years	-10 Years	-20 Years
AAA	0.9996	0.9979	0.9954	0.9985	0.9987	0.9981	0.9966	0.9984	0.9837	0.9919
AA	0.9996	0.9979	0.9954	0.9985	0.9987	0.9981	0.9965	0.9983	0.9782	1.0006
A	0.9996	0.9979	0.9954	0.9985	0.9988	0.9983	0.9973	0.9988	1.0002	1.0039
BBB	0.9911	0.9967	0.9917	0.9979	0.9981	0.9979	0.9989	1.0025	1.0042	1.0264
BB	0.9266	0.9831	0.9738	0.9855	0.9812	0.9797	0.9587	0.9911	0.6213	0.6842
B	0.8390	0.9855	1.0080	0.9572	0.9854	0.9795	0.9511	1.0438	0.0010	0.0010
C	0.7607	1.0227	1.0218	1.1031	1.1604	1.1124	0.9392	0.8007	0.0010	0.0010

Table 32: Risk adjustment factors economy state  $B$ 

	0 Months	1 Month	3 Months	6 Months	1 Year	2 Years	3 Years	5 Years	7 Years	10 Years
	-1 Month	-3 Months	-6 Months	-1 Year	-2 Years	-3 Years	-5 Years	-7 Years	-10 Years	-20 Years
AAA	1.0000	1.0000	0.9997	0.9932	0.9915	0.9974	0.9963	0.9996	0.9960	1.0022
AA	0.9989	0.9952	0.9918	0.9822	0.9885	0.9956	0.9893	0.9998	0.9819	1.0075
A	0.9970	0.9936	1.0003	0.9679	0.9891	0.9964	0.9939	1.0038	0.9935	1.0643
BBB	0.9775	0.9791	0.9777	0.9766	0.9901	0.9950	0.9836	1.0081	0.9890	1.1600
BB	0.9353	0.9636	0.9609	0.9783	1.0001	1.0029	1.0039	1.0436	1.0897	1.6183
B	0.8471	0.9467	0.9637	1.0017	1.0630	1.0572	1.1016	1.2141	1.0602	0.8352
CCC/C	0.7865	0.9751	1.0874	1.2833	1.7106	1.6546	2.0525	2.6400	0.0010	0.0010









